INFINITESIMAL DEFORMATIONS OF SYMMETRIC SIMPLE MODULAR LIE ALGEBRAS AND LIE SUPERALGEBRAS

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ABSTRACT. Over algebraically closed fields of positive characteristic, infinitesimal deformations of simple finite dimensional symmetric (the ones that with every root have its opposite of the same multiplicity) Lie algebras and Lie superalgebras are described for small ranks. The results are obtained by means of the Mathematica based code SuperLie.

The infinitesimal deformation given by any odd cocycle is integrable. The moduli of the deformations form, in general, a supervariety. Not each even cocycle is integrable; but for those that are integrable, the global deforms (the results of deformations) are linear with respect to the parameter.

In characteristic 2, the simple 3-dimensional Lie algebra admits a parametric family of non-isomorphic simple deforms.

Some of Shen's "variations of G(2) theme" are interpreted as two global deforms corresponding to the several of the 20 infinitesimal deforms first found by Chebochko; we give their explicit form.

1. Introduction

Hereafter, $\mathbb{K}$ is an algebraically closed field of characteristic $p > 0$ and $\mathfrak{g}$ is a finite dimensional Lie (super)algebra; $\mathbb{Z}_+$ is the set of non-negative integers.

1.1. The quest for simple modular Lie algebras. The Kostrikin–Shafarevich conjecture. Shafarevich, together with his student Kostrikin, first considered the *restricted* simple modular Lie algebras. This self-restriction was, perhaps, occasioned by the fact that only restricted Lie algebras correspond to algebraic groups, and being a geometer, albeit an algebraic one, Shafarevich did not see much reason to consider non-restricted Lie algebras. Recently, on a different occasion, Deligne gave us an advice [LL] which we interpret as a suggestion to look, if $p > 0$, at the groups (geometry) rather than at Lie algebras. Since only restricted Lie algebras correspond to algebraic groups, we interpret this advice as a certain natural restriction of the classification problem of simple Lie (super)algebras which makes the problem more tangible but requires to select deforms (the results of deformations) with p-structure as precious gems.

Be as it may, in late 1960s, Kostrikin and Shafarevich formulated a conjecture describing all simple finite dimensional (not only restricted) modular Lie algebras over algebraically closed fields of characteristic $p > 7$. The statement of the conjecture turned out to hold for $p = 7$ as well.

After 30 years of work of several teams of researchers, Block, Wilson, Premet and Strade not only proved the KSh-conjecture, but even completed the classification of all simple finite dimensional Lie algebra over algebraically closed fields of characteristic $p > 3$, see [S].

It turns out that, for $p > 3$, the classification can be formulated as follows (for details and definitions, see [BGL, Lt]):

(1)
> *Take any finite dimensional Lie algebra of the form $\mathfrak{g}(A)$ with indecomposable Cartan matrix A, or a trivial central extension $\mathfrak{c}(\mathfrak{g}(A))$ thereof. In one of the $\mathbb{Z}$-gradings of $\mathfrak{g} := \mathfrak{g}(A)$ or $\mathfrak{c}(\mathfrak{g}(A))$, take the Cartan-Tanaka-Shchepochkina (CTS) prolong $(\mathfrak{g}_-, \mathfrak{g}_0)_{*,\underline{N}}$ of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ whose dimension is bounded by a shearing parameter $\underline{N}$. All simple finite dimensional modular Lie algebras are among such prolongs or the quotients of their derived algebras modulo the center, and deformations thereof.*

Although for $p = 2, 3$, and for Lie superalgebras, certain other types of pairs $(\mathfrak{g}_-, \mathfrak{g}_0)$ also yield simple examples, the importance of the algebras of the form $\mathfrak{g}(A)$ is clear; it is also of independent interest.

Weisfeiler and Kac [WK] gave a classification of finite dimensional Lie algebras $\mathfrak{g}(A)$ with indecomposable Cartan matrix A for any $p > 0$, but although the idea of their proof is OK, the paper has several gaps and vague notions (the Brown algebra $\mathfrak{br}(3)$ was missed [Br3, KWK]; the definition of the Lie algebra with Cartan matrix given in [K] (and applicable to Lie *super*algebras and modular Lie (super)algebras) was not properly developed at the time [WK] was written; the algebras $\mathfrak{g}(A)^{(1)}/\mathfrak{c}$ — which have no Cartan matrix — are sometimes identified as having one, and so on). All these notions, and several more, are clarified in [BGL, LCh].

There were also known several scattered examples of serial and (conjecturally, in the absence of a classification) exceptional simple modular Lie algebras for $p = 3$ and 2, see [Lt]. For all these cases, there remained the problem of description of deformations. In the latest paper on the topic, with difficult results nicely explained, Natasha Chebochko gave an overview of the situation for Lie algebras of the form $\mathfrak{g}(A)$ and $\mathfrak{g}(A)^{(1)}/\mathfrak{c}$. She writes:

(2)
> "According to [Dz], for $p = 3$, the Lie algebra C_2 is the only algebra among the series A_n, B_n, C_n, D_n that admits non-trivial deformations. In [Ru] it was established that over a field of characteristic $p > 3$ all the classical Lie superalgebras are rigid. In [KuCh] and [KKCh] a new scheme was proposed for studying rigidity, and it was proved that the **classical** Lie algebras of all types over a field of characteristic $p > 2$ are rigid, except for the Lie algebra of type C_2 for $p = 3$. ("Classical" here means any algebra with indecomposable Cartan matrix (or its quotient modulo center), except Brown algebras: for $\mathfrak{br}(2)$ deformations were known, for $\mathfrak{br}(3)$ the question was open. This makes our answer concerning the Brown algebras $\mathfrak{br}(3)$ even more interesting. BGL.)
>
> For $p = 2$, some deformations of the Lie algebra of type G_2 were constructed in [She1]. ...
> The Lie algebras of type A_l for $l+1 \equiv 0 \pmod{2}$, D_l and E_7, have non-trivial centers. We shall say that the corresponding quotient algebras are of type $\overline{A_l}$, $\overline{D_l}$ and $\overline{E_7}$, respectively. ...

Theorem. *Let L be a Lie algebra over a field of characteristic 2.*
(1) If L is one of the types A_l for $l > 1$, D_l for $l \equiv 1 \pmod 2$, E_6, E_7, E_8,
$\overline{A_l}$ *for $l \neq 3, 5$, or $\overline{D_l}$ for $l \equiv 0 \pmod 2$ and $l \neq 6$, then $H^2(L; L) = 0$.*
(2) If L is of type $\overline{A_l}$ for $l = 3, 5$, then $\dim H^2(L; L) = 20$.

(3)   *(3) If L is of type D_4, then $\dim H^2(L; L) = 24$.*
(4) If L is of type D_l for $l \equiv 0 \pmod 2$ and $l > 4$ or $\overline{D_l}$ for $l \equiv 1 \pmod 2$,
then $\dim H^2(L; L) = 2l$.
(5) If L is of type $\overline{D_6}$, then $\dim H^2(L; L) = 64$.
(6) If L is of type $\overline{E_7}$, then $\dim H^2(L; L) = 56$."

1.1.1. Elucidating certain moments unclear to us in the conventional presentations. 1) The term "classical" in the above quotation (and almost all other papers and books on the topic) is applied to simple Lie algebras with Cartan matrix of the same types that exist for $p = 0$; their quotients modulo center are also "classical", but, for $p = 2$, the simple Lie algebras of Brown, Weisfeiler and Kac, and $\mathfrak{o}(2n + 1)$ are left out. Clearly, the nomenclature should be improved; Shen [She2] indicates other reasons for being unhappy with it.

Neither the $p = 2$ analogs of $\mathfrak{o}_I(2n)$ nor its simple subquotient have Cartan matrix. In the above quotation, they are not qualified as "classical", whereas we are sure they should be. So we have to consider

(4) $\mathfrak{o}_I(2n)$, $\mathfrak{o}(2n + 1)$ (a.k.a. B_l), the Brown algebras, and the Weisfeiler-Kac algebras.

2) The simple Lie algebras constitute a natural "core", but some of their relatives are "better" (are restricted, possess Cartan matrix, and are rigid, whereas their simple cores may lack one or all of these features). In particular, the abundance of deformations might point at an inner "flaw" of the object whose relative's behavior is impeccable ...

Together with Lebedev, in [BGL] and in chapters in [LCh] due to Lebedev, we clarified the above moments (we will reproduce the needed definitions in what follows).

1.1.2. Explicit cocycles v. ecology. Having written the first 15 pages of this paper we were appalled by the amount of the space saturated by explicit cocycles. Who needs them?! Let us give just the dimensions and save paper! However, it was only thanks to the explicit form of the cocycles that we were able to interpret some of the mysterious Shen's "variations". And only having the explicit form of the cocycle can one check if the local deformation is integrable or be able to compute the global one.

1.2. Main result. In [BGL], we have classified finite dimensional modular Lie superalgebras of the form $\mathfrak{g}(A)$ with indecomposable symmetrizable Cartan matrix A over algebraically closed fields, in particular, the simple Lie superalgebras of the form $\mathfrak{g}(A)$ and those for which $\mathfrak{g}(A)^{(1)}/\mathfrak{c}$, where $\mathfrak{c}$ is the center, is simple. For brevity, we will sometimes address the latter algebras also as having Cartan matrix, though, strictly speaking, they do not possess any Cartan matrix, see Warning in [BGL]. In [Ch], Chebochko described infinitesimal deformations of the "classical" Lie algebras with Cartan matrix, except for (4) for which it remained an open problem.

The Lie superalgebras with Cartan matrix, and the queer algebras $\mathfrak{psq}(n)$ for $p \neq 2$ are "*symmetric*":

(5)   With each root α they have a root $-\alpha$ of the same multiplicity.

(If $p = 2$, there are more examples of queer type Lie superalgebras, serial and exceptional, and some of them are not symmetric, see [LCh].)

In this note, a sequel to [BGL], we describe deformations of "symmetric" Lie (super)algebras:

(1) the "left out" cases (4), and of Lie superalgebras of small rank, both with Cartan matrix (and relatives thereof) classified in [BGL],

(2) of the so-called queer type,

(3) of certain Lie algebras without Cartan matrix: $\mathfrak{o}_I(2n)$, leaving its super versions without Cartan matrix ($\mathfrak{oo}_{II}(2n+1|2m+1)$, $\mathfrak{oo}_{II}(2n|2m)$, $\mathfrak{oo}_{III}(2n|2m)$, $\mathfrak{oo}_{II}(2n+1|2m)$) for a time being; we intend to do this elsewhere.

Conjecturally, the symmetric algebras of higher ranks over fields of characteristic distinct from 2 are rigid, except for the simple relatives of Lie (super)algebras of Hamiltonian vector fields; for the computers available to us at the moment, the problem is out of reach even with this code.

1.2.1. On Shen's "variations". Trying to append the results of Chebochko [Ch] by deforms of the Lie algebras she did not consider but which we consider no less "classical" than the ones she considered, we obtained, as a byproduct, an elucidation of Shen's "variations" [She1, LLg]. Shen described seven "variations" $V_iG(2)$ of $\mathfrak{g}(2)$ and three more "variations" of $\mathfrak{sl}(3)$; Shen claimed that all his examples are simple and all but two ($V_1G(2) := \mathfrak{g}(2)$ and $V_7G(2) := \mathfrak{wk}(3;a)$) are new. It was later found that the "variations" of $\mathfrak{sl}(3)$ are isomorphic to $\mathfrak{sl}(3)$, the variations of dimension 15 are not simple (this has to be verified). At the moment, the only verified claim on simple and really new Lie algebras Shen has discovered concerns the 2-parameter deformations of $\mathfrak{g}(2) \simeq \mathfrak{psl}(4)$. Several "variations" are depicted with typos: as stated, these algebras do not satisfy Jacobi identity or have ideals. Unfortunately, we did not guess how to amend the multiplication tables whereas our letters to Shen or his students remain unanswered.

Chebochko [Ch] has found all infinitesimal deformations of $\mathfrak{g}(2) \simeq \mathfrak{psl}(4)$ and we have found that each individual cocycle is integrable. In [Ch1], Chebochko found out that the space of cocycles constitutes two (apart from the origin) orbits relative $\mathrm{Aut}(\mathfrak{psl}(4))$, and described the two respective Lie algebras.

We only consider the Lie algebras of small rank because our `Mathematica`-based code `SuperLie` (see [Gr]) with which we got our results is incapable to process higher ranks on computers available to us. However, since the higher the rank the more rigid simple Lie (super)algebras are, we make the following

Conjecture. *In this paper we have found all (except for the analogs of the cases of Chebochko's theorem (3) that we have to add) the infinitesimal deformations of finite dimensional Lie algebras and Lie superalgebras with indecomposable Cartan matrices for $p = 5$ and 3 (and their simple subquotients) as well as of queer Lie superalgebras (and their simple subquotients).*

1.3. Motivations. The classification of simple finite dimensional Lie algebras $\mathfrak{g}$ over fields $\mathbb{K}$ of characteristic $p > 3$ — a proof of the generalized Kostrikin-Shafarevich conjecture — is now completed, see [S]. So it is time now to consider the cases $p = 3$, 2, and the super case, especially in view of the hidden supersymmetry of the non-super $p = 2$ case, see [ILL]). Let us list the results known earlier.

Several examples of simple Lie algebras over fields $\mathbb{K}$ for $p = 3$ not embraced by the approach of the Kostrikin-Shafarevich conjecture are given in [S]; in [GL4], some of these examples, earlier considered "mysterious", are described as (generalized) Cartan prolongs provided they are $\mathbb{Z}$-graded; other examples from [S] are deformations of these $\mathbb{Z}$-graded ones. In [WK], the simple modular Lie algebras possessing Cartan matrix are classified, some of them form parametric families, but no complete study of deformations was performed.

It was clear since long ago that the smaller characteristic, the less rigid the simple Lie algebras are; Rudakov gave an example of a 3-parameter family of simple Lie algebras (for the first published description, see [Kos]; for an exposition of the initial (never published) Rudakov's approach — in terms of the Cartan prolongation, — see [GL4]).

After Rudakov's example became known, Kostrikin and Dzhumadildaev ([DK, Dz1, Dz2, Dz3]) studied various (e.g., filtered and infinitesimal) deformations of simple vectorial Lie algebras (sometimes dubbed algebras of Cartan type); for a detailed summary of the part of their results with understandable proofs, and some new results (all pertaining to the infinitesimal deformations), see [Vi].

Rudakov's paper [Ru] clearly showed that speaking about deformations it is unnatural to consider the modular Lie algebras naively, as vector spaces: Lie algebras should be viewed as algebras in the category of varieties. This approach should, actually, be applied even over fields of characteristic 0, but the simplicity of the situation with finite dimensional Lie algebras obscures this (rather obvious) fact. Lie superalgebras can not be treated otherwise in various problems of interest in applications, see [LSh].

Recently the deformations of the simple modular Lie algebras with Cartan matrix, except for the Brown algebras we consider, were classified ([Ch, KK, KuCh, KKCh]). The findings of Chebochko did not, however, elucidate several of mysterious Shen's "variations", see [She1, LLg].

1.4. Disclaimer. 1) Same as practically everybody ([Ch, KK, KuCh, KKCh, Vi]) we disregard, for the moment, the danger of ignoring singular global deformations = an interesting but dangerous possibility pointed at by D. B. Fuchs with co-authors [FF, FL]. This danger, however, may only concern even deformations whereas most of the deformations we list here are odd.

2) In [Ch, KK, KuCh, KKCh], a complete description of deformations of the "classical" simple Lie algebras with Cartan matrix is performed over fields $\mathbb{K}$ for $p = 3$ and 2. Although in [LL] there are given reasons [1] to doubt the possibility to liberally apply the conventional, so far, definition of Lie algebra (co)homology ([Fu]) to the case where $p > 0$, these doubts seem to be inapplicable for $H^2(\mathfrak{g}; M)$ if $\mathfrak{g}$ possesses Cartan matrix, $\dim M < \infty$, and $p \neq 2$. In particular, it seems safe to compute $H^2(\mathfrak{g}; \mathfrak{g})$ and interpret the result in terms of infinitesimal deformations.

Let us explain how definition of $U(\mathfrak{g})$ affects the definition of cohomology and the notion of (co)induced representations. The scientific definition of the Lie (super)algebra cohomology is

$$(6) \qquad H^i(\mathfrak{g}; M) := \mathrm{Ext}^i_{U(\mathfrak{g})}(\mathbb{K}; M).$$

So it is clear, actually, how to *approach* the problem, at least for the modular Lie algebras obtained by means of the Kostrikin-Shafarevich approach (and its super analog): Speaking about non-super cases, take any book (e.g., [St]) in which a convenient $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ is described for any simple complex $\mathfrak{g}$, and introduce $\underline{N}$ (similar to the $\underline{N}$ in the definition of the algebra of divided powers $\mathcal{O}(n; \underline{N})$) by setting something like

$$(7) \qquad \begin{aligned} U(\mathfrak{g}; \underline{N}) := & \quad \text{subalgebra of } U_{\mathbb{Z}}(\mathfrak{g}) \text{ constructed} \\ & \quad \text{"similarly to the algebra of divided powers" } \mathcal{O}(n; \underline{N}); \\ H^i_{\underline{N}}(\mathfrak{g}; M) := & \quad \mathrm{Ext}^i_{U(\mathfrak{g}; \underline{N})}(\mathbb{K}; M). \end{aligned}$$

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[1] The overwhelming abundance of cocycles found in [Ch, Vi], is hardly a reason as we thought before we have read [Ch, Ch1]: Although there are many nontrivial cocycles, non-isomorphic deforms of $\mathfrak{g}$ correspond only to the distinct orbits in the space of cocycles under the action of the Chevalley group $\mathrm{Aut}(\mathfrak{g})$ and there are very few such orbits.

How to perform this "similar construction" of "something like" is the whole point.

Absolutely correct — in terms of the conventional definition (6) — computations of Dzhumadildaev elucidated in [Vi] imply that $\mathfrak{vect}(n; \underline{N}) := \mathfrak{der}(\mathcal{O}(n; \underline{N}))$, where only "special derivatives" are considered, is not rigid; more precisely, $\mathfrak{vect}(n; \underline{N})$ has *infinitesimal* deformations. We find this result "ideologically wrong" and believe that the cause is buried in the definitions used. Recall our arguments in favor of rigidity of $\mathfrak{vect}$, see [LL]:

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. For any $\mathfrak{h}$-module V, we define a series of coinduced $\mathfrak{g}$-modules:

$$(8) \qquad \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V; \underline{N}) := \mathrm{Hom}_{U(\mathfrak{h}; \underline{N})}(U(\mathfrak{g}; \underline{N}), V);$$

Then, in terms of the conjectural definition (7), we should have the following analog of the well-known isomorphism:

$$(9) \qquad H_{\underline{N}}^i(\mathfrak{g}; \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V; \underline{N})) \simeq H^i(\mathfrak{h}; V).$$

This isomorphism would imply that, for $\mathfrak{g} = \mathfrak{vect}(n; \underline{N})$, we should have

$$H_{\underline{N}}^2(\mathfrak{g}; \mathfrak{g}) \simeq H^2(\mathfrak{gl}(V); V) = 0, \ \text{where } \dim V = n,$$

at least, if n is not divisible by p.

The situation is opposite, in a sense, to that with the Kac-Moody groups that "did not exist" until a correct definition of cohomology was used; or with Dirac's δ-function which is not a function in the conventional sense.

Dzhumadildaev [Dz2] (also Farnsteiner and Strade [FS]) showed that for $p > 0$ the conventional analog of the statement (9), a.k.a. *Shapiro's lemma*, should be formulated differently because $H^i(\mathfrak{g}; \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(V))$ strictly contains $H^i(\mathfrak{h}; V)$. We hope that one can get rid of these extra cocycles in an appropriate theory.

However, even if we are wrong here and any reasonable theory contains these infinitesimal deformations, and nobody, as far as we know, investigated their number after factorization by the action of the corresponding automorphism group.

3) The Lie (super)algebra (co)homology can also be defined "naively", as generalizations (and dualizations) of the de Rham complex. In this approach the enveloping algebra does not appear explicitly and the divided powers we tried to introduce in item 2) of Disclaimer seem to disappear.

The divided powers of (co)chains naturally appear in the study of Lie **super**algebras for any p, even for $p = 0$, but their meaning is unclear at the moment. We suggest to denote the corresponding spaces of homology by $DPH_{(n)}^{\underline{N}}(\mathfrak{g}; M)$ and cohomology by $DPH_{\underline{N}}^{(n)}(\mathfrak{g}; M)$.

For $p = 2$ and Lie **super**algebras, there is an interpretation: Such divided (co)chains (for $\underline{N}_i > 1$ for every odd basis element $X_i \in \mathfrak{g}$) are indispensable in the study of relations and deformations of the Lie **super**algebras because the bracket is now determined by the squaring.

We denote the product of divided powers of cochains by wedge, the usual powers are denoted $(dx)^{\wedge n}$ whereas the divided powers are denoted $(dx)^{(\wedge n)}$. Actually, we should not write $dx \in \Pi(\mathfrak{g}^*)$ but rather x^* either for brevity or having in mind the element of $\mathfrak{g}^*$ unless the wedge product is needed.

4) **On integrability**. We compute infinitesimal deformations, i.e., we compute $H^2(\mathfrak{g}; \mathfrak{g})$. However, if the cocycle is odd, it certainly extends to a global deformation, albeit with an odd parameter.

Kostrikin and Dzhumadildaev ([DK]) claimed that every local (meaning infinitesimal) deformation of the Lie algebra $W_1(m)$ (we denote this algebra by $\mathfrak{vect}(1; \underline{m})$) is integrable. [2]

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[2] A question arises: in which sense is the classification of [S] complete?! It turns out: "as stated": for $p > 3$, ALL simple finite dimensional algebras are classified because although there are not only filtered

We were unable to follow their proof (reduction to a paper of R. Amayo [Proc. London Math. Soc. (3) 33 (1976), no. 1, 28–64; MR0409573 (53 #13327a) ,b]) but, to our incredulity, have encountered a similar phenomenon with most of the cocycles unearthed in this paper: Most of the infinitesimal deformations of some of the Lie (super)algebras considered here are integrable; moreover, the global deformation corresponding to a given (homogeneous with respect to weight) cocycle c is often **linear** in parameter, i.e., it is of the form

$$(10) \qquad\qquad [x, y]_{\text{new}} = [x, y] + \alpha c(x, y).$$

The deformation of $\mathfrak{osp}(4|2)$ and Shen's deformations of $\mathfrak{psl}(4)$ are linear in parameter, and so the corresponding cocycles give the global deformations. These facts encouraged us to conjecture that ALL cocycles we have found are integrable (this is not so) and directly investigate which of the other cocycles determine the new bracket (10).

We also have to consider deformations of the deforms (we intend to do this elsewhere).

1.4.1. What $\mathfrak{g}(A)$ is. The Lie (super)algebras of the form $\mathfrak{g}(A)$ (sometimes called, together with (super)algebras of certain other types, "contragredient") are determined as follows. Let $A = (A_{ij})$ be an arbitrary $n \times n$ matrix of rank l with entries in $\mathbb{K}$. Fix a vector space $\mathfrak{h}$ of dimension $2n - l$ and its dual $\mathfrak{h}^*$, select n linearly independent vectors $h_i \in \mathfrak{h}$ and n linearly independent vectors $\alpha_j \in \mathfrak{h}^*$ so that $\alpha_i(h_j) = A_{ij}$.

Let $I = \{i_1, \ldots, i_n\} \subset (\mathbb{Z}/2\mathbb{Z})^n$; consider the free Lie superalgebra $\widetilde{\mathfrak{g}}(A, I)$ generated by $e_1^{\pm}, \ldots, e_n^{\pm}$, where $p(e_j^{\pm}) = i_j$, and $\mathfrak{h}$, subject to relations (here either all superscripts $\pm$ are $+$ or all are $-$)

$$(11) \qquad [e_i^+, e_j^-] = \delta_{ij} h_i; \quad [h, e_j^{\pm}] = \pm\alpha_j(h)e_j^{\pm} \text{ for any } h \in \mathfrak{h} \text{ and any } i, j; \quad [\mathfrak{h}, \mathfrak{h}] = 0.$$

Let $\tilde{\mathfrak{g}}_{\pm}$ be subalgebras generated by $e_i^{\pm}$.

1.4.1a. Statement. *There exists a maximal ideal $\mathfrak{r}$ among the ideals of $\tilde{\mathfrak{g}}(A, I)$ whose intersection with $\mathfrak{h}$ is zero, and $\mathfrak{r}$ is the direct sum of the ideals $\mathfrak{r} \bigcap \tilde{\mathfrak{g}}_+$ and $\mathfrak{r} \bigcap \tilde{\mathfrak{g}}_-$.*

The statement is well known over $\mathbb{C}$ for the finite dimensional and certain infinite dimensional simple Lie algebras [K]; for an explicit description of the ideal $\mathfrak{r}$ for simple Lie superalgebras over $\mathbb{C}$ of the same type, see [GL1]; for the modular case, see Lebedev's chapters in [LCh].

The change

$$A \longmapsto \tilde{A} := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)A, \text{ where } \lambda_1, \ldots, \lambda_n \in \mathbb{K} \setminus \{0\},$$

preserves the Lie (super)algebra $\tilde{\mathfrak{g}}(A, I)$. The Cartan matrix A is said to be *normalized* if $A_{jj} = 0$ or 1, or 2 (only if $i_j = \bar{0}$, where $\bar{0}$ and $\bar{1}$ are residue classes modulo 2); in order to distinguish the cases $i_j = \bar{0}$ from $i_j = \bar{1}$, we write $A_{jj} = \bar{0}$ or $\bar{1}$, instead of 0 or 1, if $i_j = \bar{0}$. We will only consider normalized Cartan matrices; for them, we can skip I.

The subalgebra $\mathfrak{h}$ diagonally acts on $\tilde{\mathfrak{g}}(A)$ and the nonzero eigenspaces with nonzero eigenvalues are called roots. In the modular case, it is more appropriate to use the following definition. The algebra $\tilde{\mathfrak{g}}(A)$, where A is of size $n \times n$, is naturally $\mathbb{Z}^n$-graded, where $\deg e_i^{\pm} = (0, \ldots, 0, \pm 1, 0, \ldots, 0)$, where ± 1 occupies the ith slot. The nonzero eigenspaces with nonzero eigenvalues in **this** grading are said to be *roots*. (Over $\mathbb{C}$, this grading is equivalent to the usual grading by weights.) For any subset $B = \{\sigma_1, \ldots, \sigma_n\} \subset R$, we set:

$$R_B^{\pm} = \{\alpha \in R \mid \alpha = \pm \sum n_i \sigma_i, \ n_i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}\}.$$

deformations of Lie algebras of vectorial type, the deforms are isomorphic to something known. Ain't it a miracle: if $p = 2$, this definitely is not so.

The subset $B \subset R_B^+$ is said to be a *system of simple roots* of R (or $\mathfrak{g}$) if $\sigma_1, \ldots, \sigma_n$ are linearly independent and $R = R_B^- \coprod R_B^+$.

Set $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$. Observe that the conditions sufficient for simplicity of Lie algebras $\mathfrak{g}(A)$ over $\mathbb{C}$ fail to ensure simplicity of modular Lie algebras $\mathfrak{g}(A)$ and Lie superalgebras $\mathfrak{g}(A)$.

1.4.1b.   Warning. For $m - n \equiv 0 \pmod{p}$ (for $m = n$ over $\mathbb{C}$), neither $\mathfrak{sl}(m|n)$ nor its simple quotient [3] $\mathfrak{psl}(m|n)$ possess Cartan matrix; it is $\mathfrak{gl}(m|n)$ which possesses it.

2. The queer series

In the text-book [Ls] it is demonstrated that there are two superizations of $\mathfrak{gl}(n)$: a naive one, $\mathfrak{gl}(n|m)$, and the "queer" one, $\mathfrak{q}(n)$ which preserves an "odd" complex structure. Recall that

$$\mathfrak{q}(n) := \left\{ X \in \mathfrak{gl}(n|n) \mid [X, J_{2n}] = 0, \text{ where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\};$$

explicitly,

$$\mathfrak{q}(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$

We set

$$\mathfrak{sq}(n) := \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ such that } \mathrm{tr}(B) = 0 \right\}.$$

Let $\mathfrak{psq}(n) := \mathfrak{sq}(n)/\mathfrak{center}$ be the projectivisation of $\mathfrak{sq}(n)$. We denote the images of the A_{ij}-elements in $\mathfrak{psq}(n)$ by a_{ij} and the images of the B_{ij}-elements by b_{ij}.

It is not difficult to show that the derived superalgebra $\mathfrak{psq}^{(1)}(n) = [\mathfrak{psq}(n), \mathfrak{psq}(n)]$ is simple and is only different from $\mathfrak{psq}(n)$ for $n = pk$, when $\mathfrak{sq}(n)$ contains $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$.

3. Results

Observe that, for our problem, it does not matter which of several Cartan matrices that a given algebra $\mathfrak{g}(A)$ possesses we take: The simplest incarnation will do.

Since our algebras are symmetric with respect to the change of the sign of the roots, it suffices to consider cocycles of only non-negative (or non-positive) degrees. We do not list cocycles of positive (negative) degrees.

We denote the positive Chevalley generators by x_i, the corresponding negative ones by the y_i.

We underline the degrees of the <u>odd</u> cocycles.

3.1.   Lemma. *For $\mathfrak{g} = \mathfrak{br}(2; \alpha)$, where $\alpha \neq 0, -1$, the Cartan matrix is*

$$(12) \qquad \begin{pmatrix} 2 & -1 \\ \alpha & 2 \end{pmatrix} \quad \text{and the basis } x_1, \ x_2, \ x_3 = [x_1, x_2], \quad x_4 = -\,\mathrm{ad}_{x_2}^2(x_1).$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ is spanned by the cocycles (13).

$$
\begin{array}{ll}
\deg = -6: & 2\alpha(1+\alpha)y_1 \otimes dx_3 \wedge dx_4 + (1+\alpha)y_3 \otimes dx_1 \wedge dx_4 + y_4 \otimes dx_1 \wedge dx_3 \\[4pt]
(13) \qquad \deg = -3: & \alpha(1+\alpha)x_1 \otimes dx_2 \wedge dx_4 + 2(1+\alpha)y_2 \otimes dx_4 \wedge dy_1 + y_4 \otimes dx_2 \wedge dy_1 \\[4pt]
\deg = 0: & h_1 \otimes dx_2 \wedge dy_2 + h_1 \otimes dx_3 \wedge dy_3 + x_1 \otimes dx_3 \wedge dy_2 - y_1 \otimes dx_2 \wedge dy_3
\end{array}
$$

---

[3] If a Lie superalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ contains the space of scalar matrices $\mathfrak{s} = \mathbb{K}1_{m|n}$, we denote $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$ the projective version of $\mathfrak{g}$.

3.2. Lemma. *For* $\mathfrak{g} = \mathfrak{br}(2) = \lim\limits_{-\frac{2}{a} \to 0} \mathfrak{br}(2, a)$, *the Cartan matrix is*

(14) $\qquad \begin{pmatrix} 2 & -1 \\ -1 & \bar{0} \end{pmatrix}$ *and the basis* $x_1,\ x_2,\ x_3 = [x_1, x_2], \quad x_4 = -\operatorname{ad}^2_{x_2}(x_1)$.

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (15).

(15)
$$
\begin{aligned}
\deg = -6: &\quad y_1 \otimes dx_3 \wedge dx_4 + y_3 \otimes dx_1 \wedge dx_4 + 2\, y_4 \otimes dx_1 \wedge dx_3; \\
\deg = -3: &\quad x_1 \otimes dx_2 \wedge dx_4 + y_2 \otimes dx_4 \wedge dy_1 + y_4 \otimes dx_2 \wedge dy_1; \\
\deg = 0: &\quad h_1 \otimes dx_2 \wedge dy_2 + h_1 \otimes dx_3 \wedge dy_3 + x_1 \otimes dx_3 \wedge dy_2 - y_1 \otimes dx_2 \wedge dy_3
\end{aligned}
$$

Comment. Since we know that $\mathfrak{br}(2; a)$ admits a 3-parameter family of deformations ([GL4]), three of these five cocycles (recall also the ones of degrees 3 and 6) must be integrable. By symmetry the algebras obtained from the cocycles of of degrees ± 3 are isomorphic and so are the algebras obtained from the cocycles of of degrees ± 6. Anyway, the ≥ 3 cocycles in the above case are expected, so all cocycles are integrable, as in [DK]. Contrariwise, the next result is not expected; and we find it a bit too much (are all these cocycle integrable?!), as in [Ch, Vi]. The application of the group $\operatorname{Aut}(\mathfrak{g})$ by Kuznetsov and Chebochko [KuCh, Ch1] makes the abundance of cocycles not so scary since only the $\operatorname{Aut}(\mathfrak{g})$-orbits matter.

3.3. Lemma. *For* $\mathfrak{g} = \mathfrak{br}(3)$, *we consider the Cartan matrix*

(16) $\qquad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \bar{0} \end{pmatrix}$ *and the basis*
$$
\begin{aligned}
&x_1,\ x_2,\ x_3; \\
&x_4 = [x_1, x_2], \quad x_5 = [x_2, x_3]; \\
&x_6 = [x_3, [x_1, x_2]], \quad x_7 = [x_3, [x_2, x_3]]; \\
&x_8 = [x_3, [x_3, [x_1, x_2]]]; \\
&x_9 = [[x_2, x_3], [x_3, [x_1, x_2]]]; \\
&x_{10} = [[x_3, [x_1, x_2]], [x_3, [x_2, x_3]]]; \\
&x_{11} = [[x_3, [x_2, x_3]], [x_3, [x_3, [x_1, x_2]]]]; \\
&x_{12} = [[x_3, [x_2, x_3]], [[x_2, x_3], [x_3, [x_1, x_2]]]]; \\
&x_{13} = [[x_3, [x_3, [x_1, x_2]]], [[x_2, x_3], [x_3, [x_1, x_2]]]].
\end{aligned}
$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (17).

(17)
$$
\begin{aligned}
\deg = -18: \quad & 2\, y_3 \otimes dx_{12} \wedge dx_{13} + 2\, y_5 \otimes dx_{11} \wedge dx_{13} + 2\, y_6 \otimes dx_{11} \wedge dx_{12} + y_7 \otimes dx_{10} \wedge dx_{13} + y_8 \otimes dx_{10} \wedge dx_{12} + \\
& y_9 \otimes dx_{10} \wedge dx_{11} + 2\, y_{10} \otimes dx_7 \wedge dx_{13} + y_{10} \otimes dx_8 \wedge dx_{12} + 2\, y_{10} \otimes dx_9 \wedge dx_{11} + 2\, y_{11} \otimes dx_5 \wedge dx_{13} + \\
& y_{11} \otimes dx_6 \wedge dx_{12} + 2\, y_{11} \otimes dx_9 \wedge dx_{10} + 2\, y_{12} \otimes dx_3 \wedge dx_{13} + y_{12} \otimes dx_6 \wedge dx_{11} + 2\, y_{12} \otimes dx_8 \wedge dx_{10} \\
& + 2\, y_{13} \otimes dx_3 \wedge dx_{12} + y_{13} \otimes dx_5 \wedge dx_{11} + 2\, y_{13} \otimes dx_7 \wedge dx_{10}; \\[4pt]
\deg = -9: \quad & 2\, x_3 \otimes dx_1 \wedge dx_{13} + x_5 \otimes dx_4 \wedge dx_{13} + x_7 \otimes dx_6 \wedge dx_{13} + y_1 \otimes dx_4 \wedge dx_{10} + 2\, y_1 \otimes dx_6 \wedge dx_9 + \\
& 2\, y_1 \otimes dx_{13} \wedge dy_3 + y_4 \otimes dx_1 \wedge dx_{10} + y_4 \otimes dx_6 \wedge dx_8 + y_4 \otimes dx_{13} \wedge dy_5 + y_6 \otimes dx_1 \wedge dx_9 + \\
& y_6 \otimes dx_4 \wedge dx_8 + 2\, y_6 \otimes dx_{13} \wedge dy_7 + 2\, y_8 \otimes dx_4 \wedge dx_6 + y_9 \otimes dx_1 \wedge dx_6 + 2\, y_{10} \otimes dx_1 \wedge dx_4 \\
& + y_{13} \otimes dx_1 \wedge dy_3 + y_{13} \otimes dx_4 \wedge dy_5 + y_{13} \otimes dx_6 \wedge dy_7; \\[4pt]
\deg = -6: \quad & 2\, x_1 \otimes dx_2 \wedge dx_{10} + 2\, x_1 \otimes dx_5 \wedge dx_9 + 2\, x_1 \otimes dx_{12} \wedge dy_3 + x_3 \otimes dx_{12} \wedge dy_1 + 2\, x_6 \otimes dx_2 \wedge dx_{12} + \\
& x_8 \otimes dx_5 \wedge dx_{12} + 2\, y_2 \otimes dx_5 \wedge dx_7 + 2\, y_2 \otimes dx_{10} \wedge dy_1 + 2\, y_2 \otimes dx_{12} \wedge dy_6 + 2\, y_5 \otimes dx_2 \wedge dx_7 + \\
& y_5 \otimes dx_9 \wedge dy_1 + 2\, y_5 \otimes dx_{12} \wedge dy_8 + y_7 \otimes dx_2 \wedge dx_5 + 2\, y_9 \otimes dx_5 \wedge dy_1 + 2\, y_{10} \otimes dx_2 \wedge dy_1 \\
& + 2\, y_{12} \otimes dx_2 \wedge dy_6 + y_{12} \otimes dx_5 \wedge dy_8 + 2\, y_{12} \otimes dy_1 \wedge dy_3; \\[4pt]
\deg = -3: \quad & 2\, x_2 \otimes dx_3 \wedge dx_7 + x_2 \otimes dx_{10} \wedge dy_4 + 2\, x_2 \otimes dx_{11} \wedge dy_6 + x_4 \otimes dx_3 \wedge dx_8 + x_4 \otimes dx_{10} \wedge dy_2 + \\
& x_4 \otimes dx_{11} \wedge dy_5 + 2\, x_5 \otimes dx_{11} \wedge dy_4 + x_6 \otimes dx_{11} \wedge dy_2 + x_9 \otimes dx_3 \wedge dx_{11} + 2\, y_3 \otimes dx_7 \wedge dy_2 + \\
& 2\, y_3 \otimes dx_8 \wedge dy_4 + 2\, y_3 \otimes dx_{11} \wedge dy_9 + 2\, y_7 \otimes dx_3 \wedge dy_2 + y_8 \otimes dx_3 \wedge dy_4 + y_{10} \otimes dy_2 \wedge dy_4 \\
& + y_{11} \otimes dx_3 \wedge dy_9 + y_{11} \otimes dy_2 \wedge dy_6 + y_{11} \otimes dy_4 \wedge dy_5;
\end{aligned}
$$

3.4. Lemma. *For* $\mathfrak{g} = \mathfrak{brj}(2; 3)$, *we consider the Cartan matrix*

(18) $\qquad \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ *and the basis*
even | odd
$$
\begin{aligned}
&|\ x_1,\ x_2, \\
&x_3 = [x_1, x_2]\ | \\
&x_4 = [x_2, x_2],\ | \\
&|\ x_5 = [x_2, [x_1, x_2]] \\
&x_6 = [[x_1, x_2], [x_2, x_2]],\ | \\
&|\ x_7 = [[x_2, x_2], [x_2, [x_1, x_2]]] \\
&x_8 = [[x_1, x_2], [[x_1, x_2], [x_2, x_2]]]\ |
\end{aligned}
$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ is spanned by the cocycles

$$
\begin{aligned}
\deg = -12: \quad & 2\,y_2 \otimes dx_7 \wedge dx_8 + y_3 \otimes dx_7 \wedge dx_7 + y_4 \otimes dx_6 \wedge dx_8 + y_5 \otimes dx_6 \wedge dx_7 + 2\,y_6 \otimes dx_4 \wedge dx_8 + \\
& 2\,y_6 \otimes dx_5 \wedge dx_7 + y_7 \otimes dx_2 \wedge dx_8 + 2\,y_7 \otimes dx_3 \wedge dx_7 + 2\,y_7 \otimes dx_5 \wedge dx_6 + 2\,y_8 \otimes dx_2 \wedge dx_7 + \\
& 2\,y_8 \otimes dx_4 \wedge dx_6
\end{aligned}
$$

(19)

$$
\begin{aligned}
\deg = -6: \quad & x_2 \otimes dx_1 \wedge dx_8 + 2\,x_4 \otimes dx_3 \wedge dx_8 + 2\,y_1 \otimes dx_1 \wedge dx_6 + 2\,y_1 \otimes dx_3 \wedge dx_5 + y_1 \otimes dx_8 \wedge dy_2 + \\
& 2\,y_3 \otimes dx_1 \wedge dx_5 + 2\,y_3 \otimes dx_8 \wedge dy_4 + y_5 \otimes dx_1 \wedge dx_3 + y_6 \otimes dx_1 \wedge dx_1 + 2\,y_8 \otimes dx_1 \wedge dy_2 + \\
& 2\,y_8 \otimes dx_3 \wedge dy_4
\end{aligned}
$$

3.5. Lemma. *For $\mathfrak{g} = \mathfrak{brj}(2;5)$, we consider the Cartan matrix*

(20) $\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ *and the basis* *even | odd*

$$
\begin{aligned}
& |\; x_1,\; x_2, \\
& x_3 = [x_1, x_2],\; x_4 = [x_2, x_2]\,,\; | \\
& |\; x_5 = [x_2, [x_1, x_2]] \\
& x_6 = [[x_1, x_2], [x_2, x_2]]\,,\; | \\
& |\; x_7 = [[x_1, x_2], [x_2, [x_1, x_2]]]\,,\; x_8 = [[x_2, x_2], [x_2, [x_1, x_2]]]\,, \\
& x_9 = [[x_1, x_2], [[x_1, x_2], [x_2, x_2]]]\; | \\
& |\; x_{10} = [[x_2, [x_1, x_2]], [[x_1, x_2], [x_2, x_2]]]
\end{aligned}
$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ is spanned by the cocycles

$$
\begin{aligned}
\underline{\deg = -15} \quad & 2\,y_1 \otimes dx_{10} \wedge dx_{10} + 3\,y_3 \otimes dx_9 \wedge dx_{10} + y_5 \otimes dx_7 \wedge dx_{10} + 3\,y_6 \otimes dx_7 \wedge dx_9 + 3\,y_7 \otimes dx_5 \wedge dx_{10} + \\
& 4\,y_7 \otimes dx_6 \wedge dx_9 + y_7 \otimes dx_7 \wedge dx_8 + 2\,y_8 \otimes dx_7 \wedge dx_7 + 4\,y_9 \otimes dx_3 \wedge dx_{10} + y_9 \otimes dx_6 \wedge dx_7 + \\
& 4\,y_{10} \otimes dx_1 \wedge dx_{10} + 4\,y_{10} \otimes dx_3 \wedge dx_9 + 2\,y_{10} \otimes dx_5 \wedge dx_7
\end{aligned}
$$

(21)

$$
\begin{aligned}
\underline{\deg = -5}: \quad & 3\,x_1 \otimes dx_2 \wedge dx_8 + 4\,x_1 \otimes dx_4 \wedge dx_6 + 3\,x_1 \otimes dx_{10} \wedge dy_1 + 4\,x_3 \otimes dx_4 \wedge dx_8 + 3\,x_7 \otimes dx_8) \wedge dx_8 + \\
& y_2 \otimes dx_8 \wedge dy_1 + 4\,y_4 \otimes dx_6 \wedge dy_1 + 2\,y_4 \otimes dx_8 \wedge dy_3 + 2\,y_6 \otimes dx_4 \wedge dy_1 + 2\,y_8 \otimes dx_2 \wedge dy_1 + \\
& 3\,y_8 \otimes dx_4 \wedge dy_3 + y_8 \otimes dx_8 \wedge dy_7 + y_{10} \otimes dy_1 \wedge dy_1
\end{aligned}
$$

3.6. Lemma. *For $\mathfrak{g} = \mathfrak{g}(1, 6)$, we consider the Cartan matrix*

(22) $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ *and the basis* *even | odd*

$$
\begin{aligned}
& x_1,\; |\; x_2,\; x_3 \\
& x_5 = [x_2, x_2]\,,\; |\; x_4 = [x_1, x_2]\,, \\
& x_6 = [x_2, x_3]\,,\; x_7 = [x_2, [x_1, x_2]]\; | \\
& x_8 = [x_3, [x_1, x_2]]\,,\; |\; x_9 = [x_3, [x_2, x_2]] \\
& x_{10} = [[x_1, x_2], [x_1, x_2]]\,,\; |\; x_{11} = [[x_1, x_2], [x_2, x_3]] \\
& |\; x_{12} = [[x_1, x_2], [x_3, [x_1, x_2]]] \\
& x_{13} = [[x_2, x_3], [x_2, [x_1, x_2]]]\; | \\
& x_{14} = [[x_2, [x_1, x_2]], [x_3, [x_1, x_2]]]\; | \\
& |\; x_{15} = [[x_3, [x_2, x_2]], [[x_1, x_2], [x_1, x_2]]] \\
& x_{16} = [[x_3, [x_2, x_2]], [[x_1, x_2], [x_3, [x_1, x_2]]]]\; |
\end{aligned}
$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ is spanned by the cocycles (23).

(23)

$$
\begin{aligned}
\underline{\deg = -12}: \quad & 2\,x_2 \otimes dx_{13} \wedge dx_{16} + 2\,x_4 \otimes dx_{14} \wedge dx_{16} + y_2 \otimes dx_8 \wedge dx_{16} + 2\,y_3 \otimes dx_{13} \wedge dx_{14} + y_4 \otimes dx_6 \wedge dx_{16} + \\
& y_6 \otimes dx_4 \wedge dx_{16} + y_6 \otimes dx_8 \wedge dx_{15} + 2\,y_6 \otimes dx_{11} \wedge dx_{14} + 2\,y_6 \otimes dx_{12} \wedge dx_{13} + y_8 \otimes dx_2 \wedge dx_{16} + \\
& y_8 \otimes dx_6 \wedge dx_{15} + y_8 \otimes dx_9 \wedge dx_{14} + y_8 \otimes dx_{11} \wedge dx_{13} + y_9 \otimes dx_8 \wedge dx_{14} + y_{11} \otimes dx_6 \wedge dx_{14} \\
& + y_{11} \otimes dx_8 \wedge dx_{13} + y_{12} \otimes dx_6 \wedge dx_{13} + 2\,y_{13} \otimes dx_3 \wedge dx_{14} + 2\,y_{13} \otimes dx_6 \wedge dx_{12} + 2\,y_{13} \otimes dx_8 \wedge dx_{11} + \\
& y_{13} \otimes dx_{16} \wedge dy_2 + 2\,y_{14} \otimes dx_3 \wedge dx_{13} + y_{14} \otimes dx_6 \wedge dx_{11} + y_{14} \otimes dx_8 \wedge dx_9 + y_{14} \otimes dx_{16} \wedge dy_4 + \\
& 2\,y_{15} \otimes dx_6 \wedge dx_8 + y_{16} \otimes dx_2 \wedge dx_8 + 2\,y_{16} \otimes dx_4 \wedge dx_6 + y_{16} \otimes dx_{13} \wedge dy_2 + 2\,y_{16} \otimes dx_{14} \wedge dy_4;
\end{aligned}
$$

$$
\begin{aligned}
\underline{\deg = -6}: \quad & x_2 \otimes (dx_7 \wedge dx_{10}) + x_3 \otimes (dx_1 \wedge dx_{14}) + x_3 \otimes (dx_4 \wedge dx_{12}) + x_3 \otimes (dx_8 \wedge dx_{10}) + 2\,x_6 \otimes (dx_1 \wedge dx_{15}) + \\
& 2\,x_6 \otimes (dx_7 \wedge dx_{12}) + x_8 \otimes (dx_{10} \wedge dx_{12}) + x_9 \otimes (dx_4 \wedge dx_{15}) + 2\,x_9 \otimes (dx_7 \wedge dx_{14}) + 2\,x_9 \otimes (dx_{10} \wedge dx_{13}) \\
& + x_{13} \otimes (dx_{10} \wedge dx_{15}) + 2\,y_1 \otimes (dx_2 \wedge dx_{10}) + 2\,y_1 \otimes (dx_4 \wedge dx_7) + y_1 \otimes (dx_{14} \wedge dy_3) + y_1 \otimes (dx_{15} \wedge dy_6) \\
& + y_2 \otimes (dx_1 \wedge dx_{10}) + 2\,y_4 \otimes (dx_1 \wedge dx_7) + y_4 \otimes (dx_{12} \wedge dy_3) + y_4 \otimes (dx_{15} \wedge dy_9) + y_7 \otimes (dx_1 \wedge dx_4) \\
& + y_7 \otimes (dx_{10} \wedge dy_2) + y_7 \otimes (dx_{12} \wedge dy_6) + 2\,y_7 \otimes (dx_{14} \wedge dy_9) + 2\,y_8 \otimes (dx_{10} \wedge dy_3) + y_{10} \otimes (dx_1 \wedge dx_2) \\
& + 2\,y_{10} \otimes (dx_7 \wedge dy_2) + 2\,y_{10} \otimes (dx_8 \wedge dy_3) + y_{10} \otimes (dx_{12} \wedge dy_8) + 2\,y_{10} \otimes (dx_{13} \wedge dy_9) + y_{10} \otimes (dx_{15} \wedge dy_{13}) \\
& + 2\,y_{12} \otimes (dx_4 \wedge dy_3) + 2\,y_{12} \otimes (dx_7 \wedge dy_6) + 2\,y_{12} \otimes (dx_{10} \wedge dy_8) + 2\,y_{13} \otimes (dx_{10} \wedge dy_9) + 2\,y_{14} \otimes (dx_1 \wedge dy_3) \\
& + y_{14} \otimes (dx_7 \wedge dy_9) + 2\,y_{15} \otimes (dx_1 \wedge dy_6) + 2\,y_{15} \otimes (dx_4 \wedge dy_9) + 2\,y_{15} \otimes (dx_{10} \wedge dy_{13});
\end{aligned}
$$

$$
\begin{aligned}
\underline{\deg = -3}: \quad & 2\,x_1 \otimes (dx_2 \wedge dx_7) + 2\,x_1 \otimes (dx_4 \wedge dx_5) + 2\,x_1 \otimes (dx_{13} \wedge dy_3) + 2\,x_1 \otimes (dx_{15} \wedge dy_8) + x_3 \otimes (dx_2 \wedge dx_9) \\
& + 2\,x_3 \otimes (dx_5 \wedge dx_6) + x_3 \otimes (dx_{13} \wedge dy_1) + 2\,x_4 \otimes (dx_5 \wedge dx_7) + x_6 \otimes (dx_5 \wedge dx_9) \\
& + 2\,x_8 \otimes (dx_7 \wedge dx_9) + 2\,x_8 \otimes (dx_{15} \wedge dy_1) + x_{12} \otimes (dx_2 \wedge dx_{15}) + x_{12} \otimes (dx_5 \wedge dx_{14}) + x_{12} \otimes (dx_7 \wedge dx_{13}) \\
& + 2\,x_{14} \otimes (dx_5 \wedge dx_{15}) + 2\,y_2 \otimes (dx_7 \wedge dy_1) + 2\,y_2 \otimes (dx_9 \wedge dy_3) + 2\,y_2 \otimes (dx_{15} \wedge dy_{12}) + y_4 \otimes (dx_5 \wedge dy_1) \\
& + y_5 \otimes (dx_4 \wedge dy_1) + 2\,y_5 \otimes (dx_6 \wedge dy_3) + y_5 \otimes (dx_7 \wedge dy_4) + 2\,y_5 \otimes (dx_9 \wedge dy_6) + y_5 \otimes (dx_{14} \wedge dy_{12}) \\
& + y_5 \otimes (dx_{15} \wedge dy_{14}) + y_6 \otimes (dx_5 \wedge dy_3) + y_7 \otimes (dx_2 \wedge dy_1) + 2\,y_7 \otimes (dx_5 \wedge dy_4) + 2\,y_7 \otimes (dx_9 \wedge dy_8) \\
& + y_7 \otimes (dx_{13} \wedge dy_{12}) + 2\,y_9 \otimes (dx_2 \wedge dy_3) + y_9 \otimes (dx_5 \wedge dy_6) + y_9 \otimes (dx_7 \wedge dy_8) + y_{13} \otimes (dx_7 \wedge dy_{12}) \\
& + 2\,y_{13} \otimes (dy_1 \wedge dy_3) + 2\,y_{14} \otimes (dx_5 \wedge dy_{12}) + 2\,y_{15} \otimes (dx_2 \wedge dy_{12}) + 2\,y_{15} \otimes (dx_5 \wedge dy_{14}) + y_{15} \otimes (dy_1 \wedge dy_8)\,.
\end{aligned}
$$

3.7. **Lemma.** *For* $\mathfrak{g} = \mathfrak{g}(2,3)$, *we consider the Cartan matrix*

(24)
$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} \quad \begin{array}{l} \textit{and the basis} \\ \textit{even} \mid \textit{odd} \end{array}$$

$$
\begin{array}{l}
x_1,\ x_2,\ \mid x_3 \\
x_4 = [x_1,\ x_2],\ \mid x_5 = [x_1,\ x_3] \\
\mid x_6 = [x_2,\ x_3],\ x_7 = [x_3,\ [x_1,\ x_2]], \\
\mid x_8 = [[x_1,\ x_2],\ [x_1,\ x_3]],\ x_9 = [[x_1,\ x_2],\ [x_2,\ x_3]] \\
\mid x_{10} = [[x_1,\ x_2],\ [x_3,\ [x_1,\ x_2]]] \\
x_{11} = [[x_2,\ x_3],\ [[x_1,\ x_2],\ [x_1,\ x_3]]]\ \mid
\end{array}
$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (25).

(25)
$$
\begin{aligned}
\underline{\deg = -9}: \quad & 2\,x_2 \otimes dx_9 \wedge dx_{11} + 2\,x_4 \otimes dx_{10} \wedge dx_{11} + y_2 \otimes dx_5 \wedge dx_{11} + y_3 \otimes dx_4 \wedge dx_{11} + 2\,y_3 \otimes dx_7 \wedge dx_{10} + \\
& y_3 \otimes dx_8 \wedge dx_9 + y_4 \otimes dx_3 \wedge dx_{11} + y_5 \otimes dx_2 \wedge dx_{11} + 2\,y_5 \otimes dx_6 \wedge dx_{10} + y_5 \otimes dx_7 \wedge dx_9 + \\
& 2\,y_6 \otimes dx_5) \wedge dx_{10} + y_7 \otimes dx_3 \wedge dx_{10} + y_7 \otimes dx_5 \wedge dx_9 + 2\,y_8 \otimes dx_3 \wedge dx_9 + 2\,y_9 \otimes dx_3 \wedge dx_8 + \\
& y_9 \otimes dx_5 \wedge dx_7 + y_9 \otimes dx_{11} \wedge dy_2 + 2\,y_{10} \otimes dx_3 \wedge dx_7 + y_{10} \otimes dx_5 \wedge dx_6 + y_{10} \otimes dx_{11} \wedge dy_4 + \\
& 2\,y_{11} \otimes dx_2 \wedge dx_5 + y_{11} \otimes dx_3 \wedge dx_4 + y_{11} \otimes dx_9 \wedge dy_2 + 2\,y_{11} \otimes dx_{10} \wedge dy_4
\end{aligned}
$$

3.8. **Lemma.** *For* $\mathfrak{g} = \mathfrak{g}(3,3)$ *and* $\mathfrak{g} = \mathfrak{g}(4,3)$, *we have* $H^2(\mathfrak{g}; \mathfrak{g}) = 0$.

3.9. **Lemma.** *The "classical" (serial) simple Lie superalgebras of rank* $= 2$ *and* 3 *and with Cartan matrix are rigid if* $p = 3$ *and* 5, *except for* $\mathfrak{osp}(4|2)$. *The corresponding cocycles are* (26) *for* $p \geq 5$ *and* $p = 0$:

(26)
$$
\begin{aligned}
& 2h_1 \otimes dx_2 \wedge dy_2 + 3h_1 \otimes dx_3 \wedge dy_3 + 2h_1 \otimes dx_4 \wedge dy_4 + h_1 \otimes dx_6 \wedge dy_6 + 3h_1 \otimes dx_7 \wedge dy_7 + \\
& h_2 \otimes dx_3 \wedge dy_3 + 4h_2 \otimes dx_5 \wedge dy_5 + h_2 \otimes dx_6 \wedge dy_6 + h_2 \otimes dx_7 \wedge dy_7 + 4x_1 \otimes dh_3 \wedge dx_1 + \\
& x_1 \otimes dx_4 \wedge dy_2 + x_1 \otimes dx_6 \wedge dy_5 + 2x_2 \otimes dh_2 \wedge dx_2 + x_2 \otimes dh_3 \wedge dx_2 + 3x_3 \otimes dh_2 \wedge dx_3 + \\
& 2x_4 \otimes dh_2 \wedge dx_4 + x_5 \otimes dh_3 \wedge dx_5 + 3x_5 \otimes dx_2 \wedge dx_3 + 2x_6 \otimes dx_3 \wedge dx_4 + 2x_7 \otimes dh_2 \wedge dx_7 + \\
& x_7 \otimes dh_3 \wedge dx_7 + 3x_7 \otimes dx_2 \wedge dx_6 + 3x_7 \otimes dx_4 \wedge dx_5 + y_1 \otimes dh_3 \wedge dy_1 + 4y_1 \otimes dx_2 \wedge dy_4 + \\
& 4y_1 \otimes dx_5 \wedge dy_6 + 3y_2 \otimes dh_2 \wedge dy_2 + 4y_2 \otimes dh_3 \wedge dy_2 + 3y_2 \otimes dx_3 \wedge dy_5 + 2y_2 \otimes dx_6 \wedge dy_7 + \\
& 2y_3 \otimes dh_2 \wedge dy_3 + 2y_3 \otimes dx_2 \wedge dy_5 + 2y_3 \otimes dx_4 \wedge dy_6 + 3y_4 \otimes dh_2 \wedge dy_4 + 2y_4 \otimes dx_3 \wedge dy_6 + \\
& 3y_4 \otimes dx_5 \wedge dy_7 + 4y_5 \otimes dh_3 \wedge dy_5 + 3y_5 \otimes dx_4 \wedge dy_7 + 2y_6 \otimes dx_2 \wedge dy_7 + 3y_7 \otimes dh_2 \wedge dy_7 + \\
& 4y_7 \otimes dh_3 \wedge dy_7
\end{aligned}
$$

and (27) *for* $p = 3$:

(27)
$$
\begin{aligned}
& h_1 \otimes dx_2 \wedge dy_2 + h_1 \otimes dx_3 \wedge dy_3 + h_1 \otimes dx_4 \wedge dy_4 + h_1 \otimes dx_5 \wedge dy_5 + \\
& h_1 \otimes dx_7 \wedge dy_7 + 2h_2 \otimes dx_3 \wedge dy_3 + h_2 \otimes dx_7 \wedge dy_7 + x_1 \otimes dh_3 \wedge dx_1 + \\
& x_1 \otimes dx_4 \wedge dy_2 + x_1 \otimes dx_6 \wedge dy_5 + x_2 \otimes dh_2 \wedge dx_2 + 2x_2 \otimes dh_3 \wedge dx_2 + \\
& 2x_2 \otimes dx_5 \wedge dy_3 + x_2 \otimes dx_7 \wedge dy_6 + 2x_3 \otimes dh_2 \wedge dx_3 + x_4 \otimes dh_2 \wedge dx_4 + \\
& x_4 \otimes dx_6 \wedge dy_3 + 2x_4 \otimes dx_7 \wedge dy_5 + 2x_5 \otimes dh_3 \wedge dx_5 + 2x_5 \otimes dx_2 \wedge dx_3 + \\
& x_6 \otimes dx_3 \wedge dx_4 + x_7 \otimes dh_2 \wedge dx_7 + 2x_7 \otimes dh_3 \wedge dx_7 + 2x_7 \otimes dx_2 \wedge dx_6 + \\
& 2x_7 \otimes dx_4 \wedge dx_5 + 2y_1 \otimes dh_3 \wedge dy_1 + 2y_1 \otimes dx_2 \wedge dy_4 + 2y_1 \otimes dx_5 \wedge dy_6 + \\
& 2y_2 \otimes dh_2 \wedge dy_2 + y_2 \otimes dh_3 \wedge dy_2 + y_2 \otimes dx_3 \wedge dy_5 + y_3 \otimes dh_2 \wedge dy_3 + \\
& y_3 \otimes dx_2 \wedge dy_5 + y_3 \otimes dx_4 \wedge dy_6 + 2y_4 \otimes dh_2 \wedge dy_4 + 2y_4 \otimes dx_3 \wedge dy_6 + \\
& y_5 \otimes dh_3 \wedge dy_5 + 2y_5 \otimes dx_4 \wedge dy_7 + y_6 \otimes dx_2 \wedge dy_7 + 2y_7 \otimes dh_2 \wedge dy_7 + y_7 \otimes dh_3 \wedge dy_7
\end{aligned}
$$

3.10. **Lemma.** *For* $\mathfrak{g} = \mathfrak{psq}(3)$, *we have* $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ *for* $p = 5$.

For $p = 3$, *when* $\mathfrak{g} = \mathfrak{psq}(3)$ *is not simple, we take the simple algebra* $\mathfrak{g}' = \mathfrak{psq}^{(1)}(3)$ *and* $H^2(\mathfrak{g}'; \mathfrak{g}')$ *is spanned by the 2-cocycle*

(28)
$$
\begin{aligned}
\underline{\deg = 0}: \quad & 2\,a_{2,2} \otimes da_{2,2} \wedge db_2 + 2\,a_{2,2} \otimes da_{3,3} \wedge db_2 + 2\,a_{2,2} \otimes da_{1,2} \wedge db_{2,1} + a_{2,2} \otimes da_{1,3} \wedge db_{3,1} + a_{2,2} \otimes da_{3,2} \wedge db_{2,3} + \\
& a_{2,2} \otimes da_{3,1} \wedge db_{1,3} + 2\,a_{3,3} \otimes da_{2,2} \wedge db_1 + 2\,a_{3,3} \otimes da_{2,2} \wedge db_2 + 2\,a_{3,3} \otimes da_{3,3} \wedge db_1 + 2\,a_{3,3} \otimes da_{3,3} \wedge db_2 + \\
& a_{3,3} \otimes da_{2,1} \wedge db_{1,2} + 2\,a_{3,3} \otimes da_{3,2} \wedge db_{2,3} + a_{3,3} \otimes da_{3,1} \wedge db_{1,3} + 2\,a_{1,2} \otimes da_{1,2} \wedge db_2) + 2\,a_{1,2} \otimes da_{3,2} \wedge db_{1,3} \\
& + a_{2,3} \otimes da_{2,3} \wedge db_2 + a_{1,3} \otimes da_{2,2} \wedge db_{1,3} + a_{1,3} \otimes da_{3,3} \wedge db_{1,3} + a_{1,3} \otimes da_{2,3} \wedge db_{1,2} + a_{1,3} \otimes da_{1,3} \wedge db_2 + \\
& 2\,a_{2,1} \otimes da_{2,1} \wedge db_1 + a_{3,2} \otimes da_{3,2} \wedge db_1 + a_{3,1} \otimes da_{2,2} \wedge db_{3,1} + a_{3,1} \otimes da_{3,3} \wedge db_{3,1} + 2\,a_{3,1} \otimes da_{3,1} \wedge db_1 + \\
& a_{3,1} \otimes da_{3,1} \wedge db_2 + 2\,b_1 \otimes db_1 \wedge db_1 + b_1 \otimes db_{1,3} \wedge db_{3,1} + b_2 \otimes db_2 \wedge db_2 + b_{1,2} \otimes da_{2,2} \wedge da_{1,2} \\
& + b_{1,2} \otimes da_{3,3} \wedge da_{1,2} + b_{1,2} \otimes da_{1,3} \wedge da_{3,2} + b_{1,2} \otimes db_{1,3} \wedge db_{3,2} + 2\,b_{2,3} \otimes da_{2,2} \wedge da_{2,3} + 2\,b_{2,3} \otimes da_{3,3} \wedge da_{2,3} + \\
& 2\,b_{2,3} \otimes da_{1,3} \wedge da_{2,1} + b_{2,3} \otimes db_1 \wedge db_{2,3} + 2\,b_{2,3} \otimes db_2 \wedge db_{2,3} + 2\,b_{1,3} \otimes da_{1,2} \wedge da_{2,3} + 2\,b_{1,3} \otimes db_1 \wedge db_{1,3} + \\
& 2\,b_{2,1} \otimes da_{2,2} \wedge da_{2,1} + 2\,b_{2,1} \otimes da_{3,3} \wedge da_{2,1} + 2\,b_{2,1} \otimes da_{2,3} \wedge da_{3,1} + b_{2,1} \otimes db_1 \wedge db_{2,1} + 2\,b_{2,1} \otimes db_2 \wedge db_{2,1} \\
& + b_{2,1} \otimes db_{2,3} \wedge db_{3,1} + b_{3,2} \otimes da_{2,2} \wedge da_{3,2} + b_{3,2} \otimes da_{3,3} \wedge da_{3,2} + b_{3,2} \otimes da_{1,2} \wedge da_{3,1} + b_{3,2} \otimes db_{1,2} \wedge db_{3,1} + \\
& 2\,b_{3,1} \otimes da_{2,1} \wedge da_{3,2}
\end{aligned}
$$

3.11. **Lemma.** *For* $\mathfrak{g} = \mathfrak{psq}(4)$, *we have* $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ *for* $p = 3$ *and for* $p = 5$.

We did not insert here the (interesting) result of deformations of $\mathfrak{psq}(2)$ because it is not simple.

4. Results for $p = 2$

If the shapes of the answers for the usual cohomology and divided power one do not differ, we only give the answer for the usual one.

4.1. Lemma. *For* $\mathfrak{g} = \mathfrak{o}^{(1)}(3)$, *the space* $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (*29*):

$$(29) \qquad \qquad \deg = -2: \quad y \otimes dh \wedge dx$$

4.2. Lemma. *For* $\mathfrak{g} = \mathfrak{oo}^{(1)}_{\Pi\Pi}(1|2)$, *the space* $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (*30*):

$$(30) \qquad \deg = -2: \quad h_1 \otimes d(x_1)^{\wedge 2} + x_1 \otimes (dx_1 \wedge dx_2) + y_2 \otimes (dx_1 \wedge dy_1) + y_2 \otimes (dx_2 \wedge dy_2)$$

4.3. Lemma. *For* $\mathfrak{g} = \mathfrak{sl}(3)$, *we have* $H^2(\mathfrak{g}; \mathfrak{g}) = 0$ *for any* p.

4.4. Lemma. *For* $\mathfrak{g} = \mathfrak{o}^{(1)}_I(5)$, *we take the Cartan matrix*

$$(31) \qquad \begin{pmatrix} \bar{1} & 1 \\ 1 & \bar{0} \end{pmatrix} \qquad \begin{array}{l} \text{and the basis} \\ x_1, x_2, x_3 = [x_1, x_2], x_4 = [x_1, [x_1, x_2]]. \end{array}$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (*32*).

$$
\begin{array}{ll}
\deg = -4: & h_1 \otimes dx_2 \wedge dx_4 + x_1 \otimes dx_3 \wedge dx_4 + y_1 \otimes dx_2 \wedge dx_3 + \\
& y_2 \otimes dh_2 \wedge dx_4 + y_3 \otimes dh_2 \wedge dx_3 + y_4 \otimes dh_2 \wedge dx_2 \\[6pt]
\deg = -2: & h_1 \otimes dx_4 \wedge dy_2 + x_2 \otimes dh_1 \wedge dx_4 + x_2 \otimes dh_2 \wedge dx_4 + x_3 \otimes dx_1 \wedge dx_4 + y_1 \otimes dh_1 \wedge dx_1 + \\
& y_1 \otimes dh_2 \wedge dx_1 + y_1 \otimes dx_3 \wedge dy_2 + y_4 \otimes dh_1 \wedge dy_2 + y_4 \otimes dh_2 \wedge dy_2 + y_4 \otimes dx_1 \wedge dy_3
\end{array}
$$
(32)

4.5. Lemma. *For* $\mathfrak{g} = \mathfrak{wk}(3; \alpha)$, *where* $\alpha \neq 0, -1$, *we take the Cartan matrix*

$$(33) \qquad \begin{pmatrix} \bar{0} & \alpha & 0 \\ \alpha & \bar{0} & 1 \\ 0 & 1 & \bar{0} \end{pmatrix} \qquad \begin{array}{l} \text{and the basis} \\ x_1, \quad x_2, \quad x_3, \\ x_4 = [x_1, x_2], \quad x_5 = [x_2, x_3], \\ x_6 = [x_3, [x_1, x_2]], \\ x_7 = [[x_1, x_2], [x_2, x_3]]. \end{array}$$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles* (*34*).

(34)
$$
\begin{array}{ll}
\deg = -6: & \alpha(1 + \alpha)y_1 \otimes dx_3 \wedge dx_7 + \alpha\, y_1 \otimes dx_5 \wedge dx_6 + \alpha(1 + \alpha)y_3 \otimes dx_1 \wedge dx_7 + \alpha^2\, y_3 \otimes dx_4 \wedge dx_6 + \\
& \alpha\, y_4 \otimes dx_3 \wedge dx_6 + \alpha\, y_5 \otimes dx_1 \wedge dx_6 + y_6 \otimes dx_1 \wedge dx_5 + \alpha\, y_6 \otimes dx_3 \wedge dx_4 + y_7 \otimes dx_1 \wedge dx_3 \\[6pt]
\deg = -4: & \alpha(1 + \alpha)x_1 \otimes dx_3 \wedge dx_7 + \alpha\, x_1 \otimes dx_5 \wedge dx_6 + \alpha\, y_2 \otimes dx_3 \wedge dx_5 + \alpha\, y_3 \otimes dx_2 \wedge dx_5 + \alpha(1 + \alpha)y_3 \otimes dx_7 \wedge dy_1 + \\
& \alpha\, y_5 \otimes dx_2 \wedge dx_3 + \alpha\, y_5 \otimes dx_6 \wedge dy_1 + y_6 \otimes dx_5 \wedge dy_1 + y_7 \otimes dx_3 \wedge dy_1 \\[6pt]
\deg = -4: & \alpha(1 + \alpha)x_3 \otimes dx_1 \wedge dx_7 + \alpha^2\, x_3 \otimes dx_4 \wedge dx_6 + \alpha\, y_1 \otimes dx_2 \wedge dx_4 + \alpha(1 + \alpha)y_1 \otimes dx_7 \wedge dy_3 + \alpha\, y_2 \otimes dx_1 \wedge dx_4 + \\
& y_4 \otimes dx_1 \wedge dx_2 + \alpha\, y_4 \otimes dx_6 \wedge dy_3 + \alpha\, y_6 \otimes dx_4 \wedge dy_3 + y_7 \otimes dx_1 \wedge dy_3 \\[6pt]
\deg = -2: & \alpha\, x_1 \otimes dx_2 \wedge dx_4 + \alpha(1 + \alpha)x_1 \otimes dx_7 \wedge dy_3 + \alpha\, x_3 \otimes dx_2 \wedge dx_5 + \alpha(1 + \alpha)x_3 \otimes dx_7 \wedge dy_1 + \alpha\, y_2 \otimes dx_4 \wedge dy_1 + \\
& \alpha\, y_2 \otimes dx_5 \wedge dy_3 + y_4 \otimes dx_2 \wedge dy_1 + \alpha\, y_5 \otimes dx_2 \wedge dy_3 + y_7 \otimes dy_1 \wedge dy_3 \\[6pt]
\deg = 0: & h_1 \otimes (dx_4 \wedge dy_4) + h_1 \otimes (dx_6 \wedge dy_6) + h_1 \otimes (dx_7 \wedge dy_7) + \alpha\, h_3 \otimes (dx_7 \wedge dy_7) + h_4 \otimes (dx_2 \wedge dy_2) \\
& + \alpha\, h_4 \otimes (dx_4 \wedge dy_4) + h_4 \otimes (dx_5 \wedge dy_5) + \alpha\, h_4 \otimes (dx_6 \wedge dy_6) + x_1 \otimes (dx_4 \wedge dy_2) + x_1 \otimes (dx_6 \wedge dy_5) \\
& + \alpha\, x_2 \otimes (dx_7 \wedge dy_6) + x_4 \otimes (dx_7 \wedge dy_5) + \alpha\, x_5 \otimes (dx_7 \wedge dy_4) + x_6 \otimes (dx_7 \wedge dy_2) + y_1 \otimes (dx_2 \wedge dy_4) \\
& + y_1 \otimes (dx_5 \wedge dy_6) + \alpha\, y_2 \otimes (dx_6 \wedge dy_7) + y_4 \otimes (dx_5 \wedge dy_7) + \alpha\, y_5 \otimes (dx_4 \wedge dy_7) + y_6 \otimes (dx_2 \wedge dy_7)
\end{array}
$$

4.6. Lemma. *For* $\mathfrak{g} = \mathfrak{bgl}(3; \alpha)$, *where* $\alpha \neq 0, -1$, *(the super analog of* $\mathfrak{wk}(3; \alpha)$ *and an analog of non-existing* $\mathfrak{osp}(4|2; \alpha)$ *for* $p = 2$), *we take the Cartan matrix*

$$(35) \qquad \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & \bar{0} & 1 \\ 0 & 1 & \bar{0} \end{pmatrix} \qquad \begin{array}{l} \text{and the basis: even | odd} \\ x_2, \quad x_3 \mid x_1, \\ x_5 = [x_2, x_3] \mid x_4 = [x_1, x_2], \\ \mid x_6 = [x_3, [x_1, x_2]], \\ \mid x_7 = [[x_1, x_2], [x_2, x_3]] \end{array}$$

Then $H^2(\mathfrak{g};\mathfrak{g})$ is spanned by the cocycles (36).

(36)

$\deg = -8:\quad h_1 \otimes dx_7^{\wedge 2} + \alpha\, h_3 \otimes dx_7^{\wedge 2}$

$\deg = -6:\quad h_1 \otimes dx_6^{\wedge 2} + \alpha\, h_3 \otimes dx_6^{\wedge 2}$

$\deg = -4:\quad \alpha(1+\alpha)x_1 \otimes dx_3 \wedge dx_7 + \alpha\, x_1 \otimes dx_5 \wedge dx_6 + \alpha\, y_2 \otimes dx_3 \wedge dx_5 + \alpha\, y_3 \otimes dx_2 \wedge dx_5 +$
$\qquad\quad \alpha(1+\alpha)y_3 \otimes dx_7 \wedge dy_1 + \alpha\, y_5 \otimes dx_2 \wedge dx_3 + \alpha\, y_5 \otimes dx_6 \wedge dy_1 + y_6 \otimes dx_5 \wedge dy_1 + y_7 \otimes dx_3 \wedge dy_1$

$\deg = -4:\quad h_1 \otimes dx_4^{\wedge 2} + \alpha\, h_3 \otimes dx_4^{\wedge 2}$

$\deg = -4:\quad \alpha^2 h_2 \otimes dx_4^{\wedge 2} + \alpha^2 h_3 \otimes dx_4^{\wedge 2} + (\alpha^3 + \alpha^2) h_4 \otimes dx_4^{\wedge 2} + (\alpha^2 + \alpha)\, x_3 \otimes (dx_1 \wedge dx_7) + \alpha^2 x_3 \otimes (dx_4 \wedge dx_6)$
$\qquad\quad + \alpha\, y_1 \otimes (dx_2 \wedge dx_4) + (\alpha^2 + \alpha)\, y_1 \otimes (dx_7 \wedge dy_3) + \alpha\, y_2 \otimes (dx_1 \wedge dx_4) + y_4 \otimes (dx_1 \wedge dx_2)$
$\qquad\quad + \alpha\, y_4 \otimes (dx_6 \wedge dy_3) + \alpha\, y_6 \otimes (dx_4 \wedge dy_3) + y_7 \otimes (dx_1 \wedge dy_3)$

$\deg = -2:\quad h_1 \otimes dx_1^{\wedge 2} + \alpha\, h_3 \otimes dx_1^{\wedge 2}$

$\deg = -2:\quad \alpha\, x_1 \otimes dx_2 \wedge dx_4 + \alpha(1+\alpha)x_1 \otimes dx_7 \wedge dy_3 + \alpha\, x_3 \otimes dx_2 \wedge dx_5 + \alpha(1+\alpha)x_3 \otimes dx_7 \wedge dy_1 +$
$\qquad\quad \alpha\, y_2 \otimes dx_4 \wedge dy_1 + \alpha\, y_2 \otimes dx_5 \wedge dy_3 + y_4 \otimes dx_2 \wedge dy_1 + \alpha\, y_5 \otimes dx_2 \wedge dy_3 + y_7 \otimes dy_1 \wedge dy_3$

$\deg = 0:\quad h_1 \otimes (dx_4 \wedge dy_4) + h_1 \otimes (dx_6 \wedge dy_6) + h_1 \otimes (dx_7 \wedge dy_7) + \alpha\, h_3 \otimes (dx_7 \wedge dy_7) + h_4 \otimes (dx_2 \wedge dy_2)$
$\qquad\quad + \alpha\, h_4 \otimes (dx_4 \wedge dy_4) + h_4 \otimes (dx_5 \wedge dy_5) + \alpha\, h_4 \otimes (dx_6 \wedge dy_6) + x_1 \otimes (dx_4 \wedge dy_2) + x_1 \otimes (dx_6 \wedge dy_5)$
$\qquad\quad + \alpha\, x_2 \otimes (dx_7 \wedge dy_6) + x_4 \otimes (dx_7 \wedge dy_5) + \alpha\, x_5 \otimes (dx_7 \wedge dy_4) + x_6 \otimes (dx_7 \wedge dy_2) + y_1 \otimes (dx_2 \wedge dy_4)$
$\qquad\quad + y_1 \otimes (dx_5 \wedge dy_6) + \alpha\, y_2 \otimes (dx_6 \wedge dy_7) + y_4 \otimes (dx_5 \wedge dy_7) + \alpha\, y_5 \otimes (dx_4 \wedge dy_7) + y_6 \otimes (dx_2 \wedge dy_7)$

4.7. Lemma. *For* $\mathfrak{g} = \mathfrak{wk}(4;\alpha)$, *where* $\alpha \neq 0, -1$, *we take the Cartan matrix and the basis*

(37)

$$\begin{pmatrix} \bar{0} & \alpha & 1 & 0 \\ \alpha & \bar{0} & 0 & 0 \\ 1 & 0 & \bar{0} & 1 \\ 0 & 0 & 1 & \bar{0} \end{pmatrix}$$

$x_1,\ x_2,\ x_3,\ x_4,$
$[x_1, x_2],\ [x_1, x_3],\ [x_3, x_4],$
$[x_3, [x_1, x_2]],\ [x_4, [x_1, x_3]],$
$[[x_1, x_2], [x_1, x_3]],\ [[x_1, x_2], [x_3, x_4]],$
$[[x_1, x_2], [x_4, [x_1, x_3]]],$
$[[x_3, [x_1, x_2]], [x_4, [x_1, x_3]]],$
$[[x_4, [x_1, x_3]], [[x_1, x_2], [x_1, x_3]]],$
$[[[x_1, x_2], [x_1, x_3]], [[x_1, x_2], [x_3, x_4]]]$

Then $H^2(\mathfrak{g};\mathfrak{g})$ *is spanned by the cocycles which for polygraphical reasons are divided into three groups:* (38), (39), (41), (40).

$\deg = -12\quad \alpha^3(1+\alpha)y_3 \otimes dx_9 \wedge dx_{15} + \alpha^3(1+\alpha)y_3 \otimes dx_{11} \wedge dx_{14} + \alpha^2(1+\alpha)y_3 \otimes dx_{12} \wedge dx_{13} +$
$\qquad\qquad \alpha^3(1+\alpha)y_6 \otimes dx_7 \wedge dx_{15} + \alpha^2(1+\alpha)y_6 \otimes dx_{11} \wedge dx_{13} + \alpha^3(1+\alpha)y_7 \otimes dx_6 \wedge dx_{15} +$
$\qquad\qquad \alpha^3(1+\alpha)y_7 \otimes dx_8 \wedge dx_{14} + \alpha^2(1+\alpha)y_7 \otimes dx_{10} \wedge dx_{13} + \alpha^2(1+\alpha)y_8 \otimes dx_7 \wedge dx_{14} +$
$\qquad\qquad \alpha(1+\alpha)y_8 \otimes dx_9 \wedge dx_{13} + \alpha^3(1+\alpha)y_9 \otimes dx_3 \wedge dx_{15} + \alpha^2(1+\alpha)y_9 \otimes dx_8 \wedge dx_{13} +$
$\qquad\qquad \alpha(1+\alpha)y_{10} \otimes dx_7 \wedge dx_{13} + \alpha^2(1+\alpha)y_{11} \otimes dx_3 \wedge dx_{14} + \alpha(1+\alpha)y_{11} \otimes dx_6 \wedge dx_{13} +$
$\qquad\qquad \alpha(1+\alpha)y_{12} \otimes dx_3 \wedge dx_{13} + \alpha(1+\alpha)y_{13} \otimes dx_3 \wedge dx_{12} + \alpha\, y_{13} \otimes dx_6 \wedge dx_{11} +$
$\qquad\qquad \alpha(1+\alpha)y_{13} \otimes dx_7 \wedge dx_{10} + \alpha\, y_{13} \otimes dx_8 \wedge dx_9 + \alpha\, y_{14} \otimes dx_3 \wedge dx_{11} +$
$\qquad\qquad \alpha\, y_{14} \otimes dx_7 \wedge dx_8 + y_{15} \otimes dx_3 \wedge dx_9 + y_{15} \otimes dx_6 \wedge dx_7$

$\deg = -10\quad \alpha^3(1+\alpha)x_3 \otimes dx_9 \wedge dx_{15} + \alpha^3(1+\alpha)x_3 \otimes dx_{11} \wedge dx_{14} + \alpha^2(1+\alpha)x_3 \otimes dx_{12} \wedge dx_{13} +$
$\qquad\qquad \alpha^3(1+\alpha)y_1 \otimes dx_4 \wedge dx_{15} + \alpha^2(1+\alpha)y_1 \otimes dx_{11} \wedge dx_{12} + \alpha^3(1+\alpha)y_4 \otimes dx_1 \wedge dx_{15} +$
$\qquad\qquad \alpha^3(1+\alpha)y_4 \otimes dx_5 \wedge dx_{14} + \alpha^2(1+\alpha)y_4 \otimes dx_{10} \wedge dx_{12} + \alpha^2(1+\alpha)y_5 \otimes dx_4 \wedge dx_{14} +$
$\qquad\qquad \alpha(1+\alpha)y_5 \otimes dx_9 \wedge dx_{12} + \alpha^2(1+\alpha)y_9 \otimes dx_5 \wedge dx_{12} + \alpha^3(1+\alpha)y_9 \otimes dx_{15} \wedge dy_3 +$
(38)$\qquad\qquad \alpha(1+\alpha)y_{10} \otimes dx_4 \wedge dx_{12} + \alpha(1+\alpha)y_{11} \otimes dx_1 \wedge dx_{12} +$
$\qquad\qquad \alpha^2(1+\alpha)y_{11} \otimes dx_{14} \wedge dy_3 + \alpha\, y_{12} \otimes dx_1 \wedge dx_{11} + \alpha(1+\alpha)y_{12} \otimes dx_4 \wedge dx_{10} +$
$\qquad\qquad \alpha\, y_{12} \otimes dx_5 \wedge dx_9 + \alpha(1+\alpha)y_{12} \otimes dx_{13} \wedge dy_3 + \alpha(1+\alpha)y_{13} \otimes dx_{12} \wedge dy_3 +$
$\qquad\qquad \alpha\, y_{14} \otimes dx_4 \wedge dx_5 + \alpha\, y_{14} \otimes dx_{11} \wedge dy_3 + y_{15} \otimes dx_1 \wedge dx_4 + y_{15} \otimes dx_9 \wedge dy_3$

$\deg = -8\quad \alpha^3(1+\alpha)x_4 \otimes dx_1 \wedge dx_{15} + \alpha^3(1+\alpha)x_4 \otimes dx_5 \wedge dx_{14} + \alpha^2(1+\alpha)x_4 \otimes dx_{10} \wedge dx_{12} +$
$\qquad\quad \alpha^3(1+\alpha)x_7 \otimes dx_6 \wedge dx_{15} + \alpha^3(1+\alpha)x_7 \otimes dx_8 \wedge dx_{14} + \alpha^3(1+\alpha)x_7 \otimes dx_{10} \wedge dx_{13} +$
$\qquad\quad \alpha^2(1+\alpha)y_1 \otimes dx_8 \wedge dx_{10} + \alpha^3(1+\alpha)y_1 \otimes dx_{15} \wedge dy_4 + \alpha(1+\alpha)y_5 \otimes dx_6 \wedge dx_{10} +$
$\qquad\quad \alpha^2(1+\alpha)y_5 \otimes dx_{14} \wedge dy_4 + \alpha^2(1+\alpha)y_6 \otimes dx_5 \wedge dx_{10} + \alpha^3(1+\alpha)y_6 \otimes dx_{15} \wedge dy_7 +$
$\qquad\quad \alpha(1+\alpha)y_8 \otimes dx_1 \wedge dx_{10} + \alpha^2(1+\alpha)y_8 \otimes dx_{14} \wedge dy_7 + \alpha\, y_{10} \otimes dx_1 \wedge dx_8 +$
$\qquad\quad \alpha\, y_{10} \otimes dx_5 \wedge dx_6 + \alpha(1+\alpha)y_{10} \otimes dx_{12} \wedge dy_4 + \alpha(1+\alpha)\, y_{10} \otimes dx_{13} \wedge dy_7 +$
$\qquad\quad \alpha(1+\alpha)y_{12} \otimes dx_{10} \wedge dy_4 + \alpha(1+\alpha)y_{13} \otimes dx_{10} \wedge dy_7 + \alpha\, y_{14} \otimes dx_5 \wedge dy_4 +$
$\qquad\quad \alpha\, y_{14} \otimes dx_8 \wedge dy_7 + y_{15} \otimes dx_1 \wedge dy_4 + y_{15} \otimes dx_6 \wedge dy_7$

$$
\begin{aligned}
\deg = -8 \quad & \alpha^3(1+\alpha)x_1 \otimes dx_4 \wedge dx_{15} + \alpha^2(1+\alpha)x_1 \otimes dx_{11} \wedge dx_{12} + \alpha^3(1+\alpha)x_6 \otimes dx_7 \wedge dx_{15} + \\
& \alpha^2(1+\alpha)x_6 \otimes dx_{11} \wedge dx_{13} + \alpha^2(1+\alpha)y_2 \otimes dx_4 \wedge dx_{13} + \alpha^2(1+\alpha)y_2 \otimes dx_7 \wedge dx_{12} + \\
& \alpha^2(1+\alpha)y_4 \otimes dx_2 \wedge dx_{13} + \alpha^2(1+\alpha)y_4 \otimes dx_8 \wedge dx_{11} + \alpha^3(1+\alpha)y_4 \otimes dx_{15} \wedge dy_1 + \\
& \alpha(1+\alpha)y_5 \otimes dx_7 \wedge dx_{11} + \alpha^2(1+\alpha)y_7 \otimes dx_2 \wedge dx_{12} + \alpha^2(1+\alpha)y_7 \otimes dx_5 \wedge dx_{11} + \\
& \alpha^3(1+\alpha)y_7 \otimes dx_{15} \wedge dy_6 + \alpha(1+\alpha)y_8 \otimes dx_4 \wedge dx_{11} + \alpha(1+\alpha)y_{11} \otimes dx_4 \wedge dx_8 + \\
& \alpha(1+\alpha)y_{11} \otimes dx_5 \wedge dx_7 + \alpha(1+\alpha)y_{11} \otimes dx_{12} \wedge dy_1 + \alpha(1+\alpha)\,y_{11} \otimes dx_{13} \wedge dy_6 + \\
& \alpha\, y_{12} \otimes dx_2 \wedge dx_7 + \alpha\, y_{12} \otimes dx_{11} \wedge dy_1 + \alpha\, y_{13} \otimes dx_2 \wedge dx_4 + \\
& \alpha\, y_{13} \otimes dx_{11} \wedge dy_6 + y_{15} \otimes dx_4 \wedge dy_1 + y_{15} \otimes dx_7 \wedge dy_6
\end{aligned}
$$

$$
\begin{aligned}
\deg = -6 \quad & \alpha(1+\alpha)x_2 \otimes dx_4 \wedge dx_{13} + \alpha(1+\alpha)x_2 \otimes dx_7 \wedge dx_{12} + \alpha(1+\alpha)x_5 \otimes dx_4 \wedge dx_{14} + \\
& (\alpha+1)\,x_5 \otimes dx_9 \wedge dx_{12} + \alpha(1+\alpha)x_8 \otimes dx_7 \wedge dx_{14} + (\alpha+1)\,x_8 \otimes dx_9 \wedge dx_{13} + \\
& (\alpha+1)\,y_1 \otimes dx_7 \wedge dx_9 + (\alpha+1)\,y_4 \otimes dx_6 \wedge dx_9 + \alpha(1+\alpha)y_4 \otimes dx_{13} \wedge dy_2 + \\
& \alpha^2(1+\alpha)y_4 \otimes dx_{14} \wedge dy_5 + (\alpha+1)\,y_6 \otimes dx_4 \wedge dx_9 + (\alpha+1)\,y_7 \otimes dx_1 \wedge dx_9 + \\
& \alpha(1+\alpha)y_7 \otimes dx_{12} \wedge dy_2 + \alpha^2(1+\alpha)y_7 \otimes dx_{14} \wedge dy_8 + (\alpha+1)\,y_9 \otimes dx_1 \wedge dx_7 + \\
& (\alpha+1)\,y_9 \otimes dx_4 \wedge dx_6 + \alpha(1+\alpha)y_9 \otimes dx_{12} \wedge dy_5 + \alpha(1+\alpha)y_9 \otimes dx_{13} \wedge dy_8 + \\
& y_{12} \otimes dx_7 \wedge dy_2 + y_{12} \otimes dx_9 \wedge dy_5 + y_{13} \otimes dx_4 \wedge dy_2 + y_{13} \otimes dx_9 \wedge dy_8 + \\
& y_{14} \otimes dx_4 \wedge dy_5 + y_{14} \otimes dx_7 \wedge dy_8
\end{aligned}
$$

(39)

$$
\begin{aligned}
\deg = -6 \quad & \alpha^2(1+\alpha)x_1 \otimes dx_8 \wedge dx_{10} + \alpha^3(1+\alpha)x_1 \otimes dx_{15} \wedge dy_4 + \alpha^2(1+\alpha)x_4 \otimes dx_2 \wedge dx_{13} + \\
& \alpha^2(1+\alpha)x_4 \otimes dx_8 \wedge dx_{11} + \alpha^3(1+\alpha)x_4 \otimes dx_{15} \wedge dy_1 + \alpha^3(1+\alpha)x_9 \otimes dx_3 \wedge dx_{15} + \\
& \alpha^2(1+\alpha)x_9 \otimes dx_8 \wedge dx_{13} + \alpha^2(1+\alpha)y_2 \otimes dx_3 \wedge dx_{10} + \alpha^2(1+\alpha)y_2 \otimes dx_{13} \wedge dy_4 + \\
& \alpha^2(1+\alpha)y_3 \otimes dx_2 \wedge dx_{10} + \alpha^2(1+\alpha)y_3 \otimes dx_5 \wedge dx_8 + \alpha^3(1+\alpha)y_3 \otimes dx_{15} \wedge dy_9 + \\
& \alpha(1+\alpha)y_5 \otimes dx_3 \wedge dx_8 + \alpha(1+\alpha)y_8 \otimes dx_3 \wedge dx_5 + \alpha(1+\alpha)y_8 \otimes dx_{10} \wedge dy_1 + \\
& \alpha(1+\alpha)y_8 \otimes dx_{11} \wedge dy_4 + \alpha(1+\alpha)y_8 \otimes dx_{13} \wedge dy_9 + \alpha\, y_{10} \otimes dx_2 \wedge dx_3 + \\
& \alpha\, y_{10} \otimes dx_8 \wedge dy_1 + \alpha(1+\alpha)y_{11} \otimes dx_8 \wedge dy_4 + \alpha\, y_{13} \otimes dx_2 \wedge dy_4 + \\
& \alpha\, y_{13} \otimes dx_8 \wedge dy_9 + y_{15} \otimes dx_3 \wedge dy_9 + y_{15} \otimes dy_1 \wedge dy_4
\end{aligned}
$$

$$
\begin{aligned}
\deg = 0 \quad & \alpha\, h_1 \otimes dx_{14} \wedge dy_{14} + \alpha^2 h_1 \otimes dx_{15} \wedge dy_{15} + h_2 \otimes dx_5 \wedge dy_5 + h_2 \otimes dx_8 \wedge dy_8 + \\
& h_2 \otimes dx_{10} \wedge dy_{10} + h_2 \otimes dx_{11} \wedge dy_{11} + h_2 \otimes dx_{12} \wedge dy_{12} + h_2 \otimes dx_{13} \wedge dy_{13} + \\
& \alpha^2 h_2 \otimes dx_{14} \wedge dy_{14} + h_3 \otimes dx_1 \wedge dy_1 + \alpha\, h_3 \otimes dx_5 \wedge dy_5 + h_3 \otimes dx_6 \wedge dy_6 + \\
& \alpha\, h_3 \otimes dx_8 \wedge dy_8 + h_3 \otimes dx_9 \wedge dy_9 + \alpha\, h_3 \otimes dx_{10} \wedge dy_{10} + \alpha\, h_3 \otimes dx_{11} \wedge dy_{11} + \\
& \alpha\, h_3 \otimes dx_{12} \wedge dy_{12} + \alpha^2(1+\alpha)h_3 \otimes dx_{14} \wedge dy_{14} + \alpha^3(1+\alpha)h_3 \otimes dx_{15} \wedge dy_{15} + \\
& \alpha\, h_4 \otimes dx_{12} \wedge dy_{12} + \alpha\, h_4 \otimes dx_{13} \wedge dy_{13} + \alpha\, h_4 \otimes dx_{14} \wedge dy_{14} + \alpha^2 h_4 \otimes dx_{15} \wedge dy_{15} + \\
& x_1 \otimes dh_1 \wedge dx_1 + \alpha\, x_1 \otimes dx_{10} \wedge dy_8 + \alpha\, x_1 \otimes dx_{12} \wedge dy_{11} + \alpha\, x_1 \otimes dx_{14} \wedge dy_{13} + \\
& x_2 \otimes dh_1 \wedge dx_2 + x_2 \otimes dx_5 \wedge dy_1 + x_2 \otimes dx_8 \wedge dy_6 + x_2 \otimes dx_{11} \wedge dy_9 + \alpha^3 x_2 \otimes dx_{15} \wedge dy_{14} + \\
& \alpha\, x_3 \otimes dx_{13} \wedge dy_{12} + x_4 \otimes dh_1 \wedge dx_4 + \alpha\, x_4 \otimes dx_{12} \wedge dy_{10} + x_5 \otimes dx_{10} \wedge dy_6 + \\
& x_5 \otimes dx_{12} \wedge dy_9 + \alpha\, x_5 \otimes dx_{15} \wedge dy_{13} + x_6 \otimes dh_1 \wedge dx_6 + \alpha\, x_6 \otimes dx_{10} \wedge dy_5 + \\
& \alpha\, x_6 \otimes dx_{13} \wedge dy_{11} + \alpha\, x_6 \otimes dx_{14} \wedge dy_{12} + x_7 \otimes dh_1 \wedge dx_7 + \alpha\, x_7 \otimes dx_{13} \wedge dy_{10} + \\
& x_8 \otimes dx_{10} \wedge dy_1 + x_8 \otimes dx_{13} \wedge dy_9 + \alpha\, x_8 \otimes dx_{15} \wedge dy_{12} + \\
& \alpha\, x_9 \otimes dx_{12} \wedge dy_5 + \alpha\, x_9 \otimes dx_{13} \wedge dy_8 + \alpha\, x_9 \otimes dx_{14} \wedge dy_{10} + \\
& x_{10} \otimes dh_1 \wedge dx_{10} + x_{10} \otimes dx_{14} \wedge dy_9 + \alpha\, x_{10} \otimes dx_{15} \wedge dy_{11} + \\
& x_{11} \otimes dh_1 \wedge dx_{11} + x_{11} \otimes dx_{12} \wedge dy_1 + x_{11} \otimes dx_{13} \wedge dy_6 + \alpha\, x_{11} \otimes dx_{15} \wedge dy_{10} + \\
& x_{12} \otimes dx_{14} \wedge dy_6 + \alpha\, x_{12} \otimes dx_{15} \wedge dy_8 + x_{13} \otimes dx_{14} \wedge dy_1 + \alpha\, x_{13} \otimes dx_{15} \wedge dy_5 + \\
& x_{14} \otimes dh_1 \wedge dx_{14} + y_1 \otimes dh_1 \wedge dy_1 + \alpha\, y_1 \otimes dx_8 \wedge dy_{10} + \alpha\, y_1 \otimes dx_{11} \wedge dy_{12} + \\
& \alpha\, y_1 \otimes dx_{13} \wedge dy_{14} + y_2 \otimes dh_1 \wedge dy_2 + y_2 \otimes dx_1 \wedge dy_5 + y_2 \otimes dx_6 \wedge dy_8 + \\
& y_2 \otimes dx_9 \wedge dy_{11} + \alpha^3 y_2 \otimes dx_{14} \wedge dy_{15} + \alpha\, y_3 \otimes dx_{12} \wedge dy_{13} + y_4 \otimes dh_1 \wedge dy_4 \\
& \alpha\, y_4 \otimes dx_{10} \wedge dy_{12} + y_5 \otimes dx_6 \wedge dy_{10} + y_5 \otimes dx_9 \wedge dy_{12} + \alpha\, y_5 \otimes dx_{13} \wedge dy_{15} + \\
& y_6 \otimes dh_1 \wedge dy_6 + \alpha\, y_6 \otimes dx_5 \wedge dy_{10} + \alpha\, y_6 \otimes dx_{11} \wedge dy_{13} + \alpha\, y_6 \otimes dx_{12} \wedge dy_{14} + \\
& y_7 \otimes dh_1 \wedge dy_7 + \alpha\, y_7 \otimes dx_{10} \wedge dy_{13} + y_8 \otimes dx_1 \wedge dy_{10} + y_8 \otimes dx_9 \wedge dy_{13} + \\
& \alpha\, y_8 \otimes dx_{12} \wedge dy_{15} + \alpha\, y_9 \otimes dx_5 \wedge dy_{12} + \alpha\, y_9 \otimes dx_8 \wedge dy_{13} + \alpha\, y_9 \otimes dx_{10} \wedge dy_{14} + \\
& y_{10} \otimes dh_1 \wedge dy_{10} + y_{10} \otimes dx_9 \wedge dy_{14} + \alpha\, y_{10} \otimes dx_{11} \wedge dy_{15} + y_{11} \otimes dh_1 \wedge dy_{11} + \\
& y_{11} \otimes dx_1 \wedge dy_{12} + y_{11} \otimes dx_6 \wedge dy_{13} + \alpha\, y_{11} \otimes dx_{10} \wedge dy_{15} + \\
& y_{12} \otimes dx_6 \wedge dy_{14} + \alpha\, y_{12} \otimes dx_8 \wedge dy_{15} + y_{13} \otimes dx_1 \wedge dy_{14} + \\
& \alpha\, y_{13} \otimes dx_5 \wedge dy_{15} + y_{14} \otimes dh_1 \wedge dy_{14}
\end{aligned}
$$

(40)

$\deg = -4$ $\quad \alpha(1+\alpha)x_2 \otimes dx_3 \wedge dx_{10} + \alpha(1+\alpha)x_2 \otimes dx_{13} \wedge dy_4 + (\alpha+1)\,x_4 \otimes dx_6 \wedge dx_9 +$
$\alpha(1+\alpha)x_4 \otimes dx_{13} \wedge dy_2 + \alpha^2(1+\alpha)x_4 \otimes dx_{14} \wedge dy_5 + (\alpha+1)\,x_5 \otimes dx_6 \wedge dx_{10} +$
$\alpha(1+\alpha)x_5 \otimes dx_{14} \wedge dy_4 + \alpha(1+\alpha)x_{11} \otimes dx_3 \wedge dx_{14} + (\alpha+1)\,x_{11} \otimes dx_6 \wedge dx_{13} +$
$(\alpha+1)\,y_1 \otimes dx_3 \wedge dx_6 + (\alpha+1)\,y_3 \otimes dx_1 \wedge dx_6 + \alpha(1+\alpha)y_3 \otimes dx_{10} \wedge dy_2 +$
$\alpha^2(1+\alpha)y_3 \otimes dx_{14} \wedge dy_{11} + (\alpha+1)\,y_6 \otimes dx_1 \wedge dx_3 + (\alpha+1)\,y_6 \otimes dx_9 \wedge dy_4 +$
$\alpha(1+\alpha)y_6 \otimes dx_{10} \wedge dy_5 + \alpha(1+\alpha)y_6 \otimes dx_{13} \wedge dy_{11} + (\alpha+1)\,y_9 \otimes dx_6 \wedge dy_4 +$
$y_{10} \otimes dx_3 \wedge dy_2 + y_{10} \otimes dx_6 \wedge dy_5 + y_{13} \otimes dx_6 \wedge dy_{11} +$
$y_{13} \otimes dy_2 \wedge dy_4 + y_{14} \otimes dx_3 \wedge dy_{11} + y_{14} \otimes dy_4 \wedge dy_5$

(41)

$\deg = -4$ $\quad \alpha^2(1+\alpha)x_3 \otimes dx_2 \wedge dx_{10} + \alpha^2(1+\alpha)x_3 \otimes dx_5 \wedge dx_8 + \alpha^3(1+\alpha)x_3 \otimes dx_{15} \wedge dy_9 +$
$\alpha^2(1+\alpha)x_6 \otimes dx_5 \wedge dx_{10} + \alpha^3(1+\alpha)x_6 \otimes dx_{15} \wedge dy_7 + \alpha^2(1+\alpha)x_7 \otimes dx_2 \wedge dx_{12} +$
$\alpha^2(1+\alpha)x_7 \otimes dx_5 \wedge dx_{11} + \alpha^3(1+\alpha)x_7 \otimes dx_{15} \wedge dy_6 + \alpha^2(1+\alpha)x_9 \otimes dx_5 \wedge dx_{12} +$
$\alpha^3(1+\alpha)x_9 \otimes dx_{15} \wedge dy_3 + \alpha^2(1+\alpha)y_2 \otimes dx_{10} \wedge dy_3 + \alpha^2(1+\alpha)y_2 \otimes dx_{12} \wedge dy_7 +$
$\alpha(1+\alpha)y_5 \otimes dx_8 \wedge dy_3 + \alpha(1+\alpha)y_5 \otimes dx_{10} \wedge dy_6 + \alpha(1+\alpha)y_5 \otimes dx_{11} \wedge dy_7 +$
$\alpha(1+\alpha)y_5 \otimes dx_{12} \wedge dy_9 + \alpha(1+\alpha)y_8 \otimes dx_5 \wedge dy_3 + \alpha\,y_{10} \otimes dx_2 \wedge dy_3 +$
$\alpha\,y_{10} \otimes dx_5 \wedge dy_6 + \alpha^2 + \alpha\,y_{11} \otimes dx_5 \wedge dy_7 + \alpha\,y_{12} \otimes dx_2 \wedge dy_7 +$
$\alpha\,y_{12} \otimes dx_5 \wedge dy_9 + y_{15} \otimes dy_3 \wedge dy_9 + y_{15} \otimes dy_6 \wedge dy_7$

$\deg = -2$ $\quad \alpha(1+\alpha)x_2 \otimes dx_{10} \wedge dy_3 + \alpha(1+\alpha)x_2 \otimes dx_{12} \wedge dy_7 + (\alpha+1)\,x_3 \otimes dx_1 \wedge dx_6 +$
$\alpha^2 + \alpha\,x_3 \otimes dx_{10} \wedge dy_2 + \alpha^3 + \alpha^2\,x_3 \otimes dx_{14} \wedge dy_{11} + (\alpha+1)\,x_7 \otimes dx_1 \wedge dx_9 +$
$\alpha(1+\alpha)x_7 \otimes dx_{12} \wedge dy_2 + \alpha^2(1+\alpha)x_7 \otimes dx_{14} \wedge dy_8 + (\alpha+1)\,x_8 \otimes dx_1 \wedge dx_{10} +$
$\alpha(1+\alpha)x_8 \otimes dx_{14} \wedge dy_7 + (\alpha+1)\,x_{11} \otimes dx_1 \wedge dx_{12} + \alpha(1+\alpha)x_{11} \otimes dx_{14} \wedge dy_3 +$
$(\alpha+1)\,y_1 \otimes dx_6 \wedge dy_3 + (\alpha+1)\,y_1 \otimes dx_9 \wedge dy_7 + \alpha^2 + \alpha\,y_1 \otimes dx_{10} \wedge dy_8 +$
$\alpha(1+\alpha)y_1 \otimes dx_{12} \wedge dy_{11} + (\alpha+1)\,y_6 \otimes dx_1 \wedge dy_3 + (\alpha+1)\,y_9 \otimes dx_1 \wedge dy_7 +$
$y_{10} \otimes dx_1 \wedge dy_8 + y_{10} \otimes dy_2 \wedge dy_3 + y_{12} \otimes dx_1 \wedge dy_{11} + y_{12} \otimes dy_2 \wedge dy_7 +$
$y_{14} \otimes dy_3 \wedge dy_{11} + y_{14} \otimes dy_7 \wedge dy_8$

4.8. Lemma. *For* $\mathfrak{g} = \mathfrak{bgl}(4;\alpha)$, *where* $\alpha \neq 0, -1$, *we take the Cartan matrix and the basis*

(42)

$$\begin{pmatrix} 0 & \alpha & 1 & 0 \\ \alpha & \bar{0} & 0 & 0 \\ 1 & 0 & \bar{0} & 1 \\ 0 & 0 & 1 & \bar{0} \end{pmatrix}$$

$x_1, \ x_2, \ x_3, \ x_4,$
$[x_1, x_2], \ [x_1, x_3], \ [x_3, x_4],$
$[x_3, [x_1, x_2]], \ [x_4, [x_1, x_3]],$
$[[x_1, x_2], [x_1, x_3]], \ [[x_1, x_2], [x_3, x_4]],$
$[[x_1, x_2], [x_4, [x_1, x_3]]],$
$[[x_3, [x_1, x_2]], [x_4, [x_1, x_3]]],$
$[[x_4, [x_1, x_3]], [[x_1, x_2], [x_1, x_3]]],$
$[[[x_1, x_2], [x_1, x_3]], [[x_1, x_2], [x_3, x_4]]]$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ *is spanned by the cocycles:*
(43)

$\deg = -12$ $\quad (\alpha^4 + \alpha^3)\, y_3 \otimes (dx_9 \wedge dx_{15}) + (\alpha^4 + \alpha^3)\, y_3 \otimes (dx_{11} \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, y_3 \otimes (dx_{12} \wedge dx_{13})$
$+ (\alpha^4 + \alpha^3)\, y_6 \otimes (dx_7 \wedge dx_{15}) + (\alpha^3 + \alpha^2)\, y_6 \otimes (dx_{11} \wedge dx_{13}) + (\alpha^4 + \alpha^3)\, y_7 \otimes (dx_6 \wedge dx_{15})$
$+ (\alpha^4 + \alpha^3)\, y_7 \otimes (dx_8 \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, y_7 \otimes (dx_{10} \wedge dx_{13}) + (\alpha^3 + \alpha^2)\, y_8 \otimes (dx_7 \wedge dx_{14})$
$+ (\alpha^2 + \alpha)\, y_8 \otimes (dx_9 \wedge dx_{13}) + (\alpha^4 + \alpha^3)\, y_9 \otimes (dx_3 \wedge dx_{15}) + (\alpha^3 + \alpha^2)\, y_9 \otimes (dx_8 \wedge dx_{13})$
$+ (\alpha^2 + \alpha)\, y_{10} \otimes (dx_7 \wedge dx_{13}) + (\alpha^3 + \alpha^2)\, y_{11} \otimes (dx_3 \wedge dx_{14}) + (\alpha^2 + \alpha)\, y_{11} \otimes (dx_6 \wedge dx_{13})$
$+ (\alpha^2 + \alpha)\, y_{12} \otimes (dx_3 \wedge dx_{13}) + (\alpha^2 + \alpha)\, y_{13} \otimes (dx_3 \wedge dx_{12}) + \alpha\, y_{13} \otimes (dx_6 \wedge dx_{11})$
$+ (\alpha^2 + \alpha)\, y_{13} \otimes (dx_7 \wedge dx_{10}) + \alpha\, y_{13} \otimes (dx_8 \wedge dx_9) + \alpha\, y_{14} \otimes (dx_3 \wedge dx_{11}) + \alpha\, y_{14} \otimes (dx_7 \wedge dx_8)$
$+ y_{15} \otimes (dx_3 \wedge dx_9) + y_{15} \otimes (dx_6 \wedge dx_7)$

$\deg = -10$ $\quad (\alpha^4 + \alpha^3)\, x_3 \otimes (dx_9 \wedge dx_{15}) + (\alpha^4 + \alpha^3)\, x_3 \otimes (dx_{11} \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, x_3 \otimes (dx_{12} \wedge dx_{13})$
$+ (\alpha^4 + \alpha^3)\, y_1 \otimes (dx_4 \wedge dx_{15}) + (\alpha^3 + \alpha^2)\, y_1 \otimes (dx_{11} \wedge dx_{12}) + (\alpha^4 + \alpha^3)\, y_4 \otimes (dx_1 \wedge dx_{15})$
$+ (\alpha^4 + \alpha^3)\, y_4 \otimes (dx_5 \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, y_4 \otimes (dx_{10} \wedge dx_{12}) + (\alpha^3 + \alpha^2)\, y_5 \otimes (dx_4 \wedge dx_{14})$
$+ (\alpha^2 + \alpha)\, y_5 \otimes (dx_9 \wedge dx_{12}) + (\alpha^3 + \alpha^2)\, y_9 \otimes (dx_5 \wedge dx_{12}) + (\alpha^4 + \alpha^3)\, y_9 \otimes (dx_{15} \wedge dy_3)$
$+ (\alpha^2 + \alpha)\, y_{10} \otimes (dx_4 \wedge dx_{12}) + (\alpha^2 + \alpha)\, y_{11} \otimes (dx_1 \wedge dx_{12}) + (\alpha^3 + \alpha^2)\, y_{11} \otimes (dx_{14} \wedge dy_3)$
$+ \alpha\, y_{12} \otimes (dx_1 \wedge dx_{11}) + (\alpha^2 + \alpha)\, y_{12} \otimes (dx_4 \wedge dx_{10}) + \alpha\, y_{12} \otimes (dx_5 \wedge dx_9)$
$+ (\alpha^2 + \alpha)\, y_{12} \otimes (dx_{13} \wedge dy_3) + (\alpha^2 + \alpha)\, y_{13} \otimes (dx_{12} \wedge dy_3) + \alpha\, y_{14} \otimes (d_4 \wedge dx_5) + \alpha\, y_{14} \otimes (dx_{11} \wedge dy_3)$
$+ y_{15} \otimes (dx_1 \wedge dx_4) + y_{15} \otimes (d_9 \wedge dy_3)$

$\deg = -8$ $\quad (\alpha^4 + \alpha^3)\, x_4 \otimes (dx_1 \wedge dx_{15}) + (\alpha^4 + \alpha^3)\, x_4 \otimes (dx_5 \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, x_4 \otimes (dx_{10} \wedge dx_{12})$
$+ (\alpha^4 + \alpha^3)\, x_7 \otimes (dx_6 \wedge dx_{15}) + (\alpha^4 + \alpha^3)\, x_7 \otimes (dx_8 \wedge dx_{14}) + (\alpha^4 + \alpha^2)\, x_7 \otimes (dx_{10} \wedge dx_{13})$
$+ (\alpha^3 + \alpha^2)\, y_1 \otimes (dx_8 \wedge dx_{10}) + (\alpha^4 + \alpha^3)\, y_1 \otimes (dx_{15} \wedge dy_4) + (\alpha^2 + \alpha)\, y_5 \otimes (dx_6 \wedge dx_{10})$
$+ (\alpha^3 + \alpha^2)\, y_5 \otimes (dx_{14} \wedge dy_4) + (\alpha^3 + \alpha^2)\, y_6 \otimes (dx_5 \wedge dx_{10}) + (\alpha^4 + \alpha^3)\, y_6 \otimes (dx_{15} \wedge dy_7)$
$+ (\alpha^2 + \alpha)\, y_8 \otimes (dx_1 \wedge dx_{10}) + (\alpha^3 + \alpha^2)\, y_8 \otimes (dx_{14} \wedge dy_7) + \alpha\, y_{10} \otimes (dx_1 \wedge dx_8)$
$+ \alpha\, y_{10} \otimes (dx_5 \wedge dx_6) + (\alpha^2 + \alpha)\, y_{10} \otimes (dx_{12} \wedge dy_4) + (\alpha^2 + \alpha)\, y_{10} \otimes (dx_{13} \wedge dy_7)$
$+ (\alpha^2 + \alpha)\, y_{12} \otimes (dx_{10} \wedge dy_4) + (\alpha^2 + \alpha)\, y_{13} \otimes (dx_{10} \wedge dy_7) + \alpha\, y_{14} \otimes (dx_5 \wedge dy_4)$
$+ \alpha\, y_{14} \otimes (dx_8 \wedge dy_7) + y_{15} \otimes (dx_1 \wedge dy_4) + y_{15} \otimes (dx_6 \wedge dy_7)$

(44)

$\deg = -6$ $(\alpha + 1)\, h_3 \otimes (dx_9)^{\wedge 2} + \left(\alpha^2 + \alpha\right) x_2 \otimes (dx_4 \wedge dx_{13}) + \left(\alpha^2 + \alpha\right) x_2 \otimes (dx_7 \wedge dx_{12})$
$+ \left(\alpha^2 + \alpha\right) x_5 \otimes (dx_4 \wedge dx_{14}) + (\alpha + 1)\, x_5 \otimes (dx_9 \wedge dx_{12}) + \left(\alpha^2 + \alpha\right) x_8 \otimes (dx_7 \wedge dx_{14})$
$+ (\alpha + 1)\, x_8 \otimes (dx_9 \wedge dx_{13}) + (\alpha + 1)\, y_1 \otimes (dx_7 \wedge dx_9) + (\alpha + 1)\, y_4 \otimes (dx_6 \wedge dx_9)$
$+ \left(\alpha^2 + \alpha\right) y_4 \otimes (dx_{13} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) y_4 \otimes (dx_{14} \wedge dy_5) + (\alpha + 1)\, y_6 \otimes (dx_4 \wedge dx_9)$
$+ (\alpha + 1)\, y_7 \otimes (dx_1 \wedge dx_9) + \left(\alpha^2 + \alpha\right) y_7 \otimes (dx_{12} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) y_7 \otimes (dx_{14} \wedge dy_8)$
$+ (\alpha + 1)\, y_9 \otimes (dx_1 \wedge dx_7) + (\alpha + 1)\, y_9 \otimes (dx_4 \wedge dx_6) + \left(\alpha^2 + \alpha\right) y_9 \otimes (dx_{12} \wedge dy_5)$
$+ \left(\alpha^2 + \alpha\right) y_9 \otimes (dx_{13} \wedge dy_8) + y_{12} \otimes (dx_7 \wedge dy_2) + y_{12} \otimes (dx_9 \wedge dy_5) + y_{13} \otimes (dx_4 \wedge dy_2)$
$+ y_{13} \otimes (dx_9 \wedge dy_8) + y_{14} \otimes (dx_4 \wedge dy_5) + y_{14} \otimes (dx_7 \wedge dy_8)$

$\deg = -6$ $\left(\alpha^3 + \alpha^2\right) x_1 \otimes (dx_8 \wedge dx_{10}) + \left(\alpha^4 + \alpha^3\right) x_1 \otimes (dx_{15} \wedge dy_4) + \left(\alpha^3 + \alpha^2\right) x_4 \otimes (dx_2 \wedge dx_{13})$
$+ \left(\alpha^3 + \alpha^2\right) x_4 \otimes (dx_8 \wedge dx_{11}) + \left(\alpha^4 + \alpha^3\right) x_4 \otimes (dx_{15} \wedge dy_1) + \left(\alpha^4 + \alpha^3\right) x_9 \otimes (dx_3 \wedge dx_{15})$
$+ \left(\alpha^3 + \alpha^2\right) x_9 \otimes (dx_8 \wedge dx_{13}) + \left(\alpha^3 + \alpha^2\right) y_2 \otimes (dx_3 \wedge dx_{10}) + \left(\alpha^3 + \alpha^2\right) y_2 \otimes (dx_{13} \wedge dy_4)$
$+ \left(\alpha^3 + \alpha^2\right) y_3 \otimes (dx_2 \wedge dx_{10}) + \left(\alpha^3 + \alpha^2\right) y_3 \otimes (dx_5 \wedge dx_8) + \left(\alpha^4 + \alpha^3\right) y_3 \otimes (dx_{15} \wedge dy_9)$
$+ \left(\alpha^2 + \alpha\right) y_5 \otimes (dx_3 \wedge dx_8) + \left(\alpha^2 + \alpha\right) y_8 \otimes (dx_3 \wedge dx_5) + \left(\alpha^2 + \alpha\right) y_8 \otimes (dx_{10} \wedge dy_1)$
$+ \left(\alpha^2 + \alpha\right) y_8 \otimes (d_{11} \wedge dy_4) + \left(\alpha^2 + \alpha\right) y_8 \otimes (dx_{13} \wedge dy_9) + \alpha\, y_{10} \otimes (dx_2 \wedge dx_3) + \alpha\, y_{10} \otimes (dx_8 \wedge dy_1)$
$+ \left(\alpha^2 + \alpha\right) y_{11} \otimes (dx_8 \wedge dy_4) + \alpha\, y_{13} \otimes (dx_2 \wedge dy_4) + \alpha\, y_{13} \otimes (dx_8 \wedge d_9) + y_{15} \otimes (dx_3 \wedge dy_9) + y_{15} \otimes (dy_1 \wedge dy_4)$

$\deg = -4$ $(\alpha + 1)\, h_3 \otimes (dx_6)^{\wedge 2} + (\alpha + 1)\, h_4 \otimes (dx_6)^{\wedge 2} + \left(\alpha^2 + \alpha\right) x_2 \otimes (dx_3 \wedge dx_{10}) + \left(\alpha^2 + \alpha\right) x_2 \otimes (dx_{13} \wedge dy_4)$
$+ (\alpha + 1)\, x_4 \otimes (dx_6 \wedge dx_9) + \left(\alpha^2 + \alpha\right) x_4 \otimes (dx_{13} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) x_4 \otimes (dx_{14} \wedge dy_5)$
$+ (\alpha + 1)\, x_5 \otimes (dx_6 \wedge dx_{10}) + \left(\alpha^2 + \alpha\right) x_5 \otimes (dx_{14} \wedge dy_4) + \left(\alpha^2 + \alpha\right) x_{11} \otimes (dx_3 \wedge dx_{14})$
$+ (\alpha + 1)\, x_{11} \otimes (dx_6 \wedge dx_{13}) + (\alpha + 1)\, y_1 \otimes (dx_3 \wedge dx_6) + (\alpha + 1)\, y_3 \otimes (dx_1 \wedge dx_6)$
$+ \left(\alpha^2 + \alpha\right) y_3 \otimes (dx_{10} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) y_3 \otimes (d_{14} \wedge dy_{11}) + (\alpha + 1)\, y_6 \otimes (dx_1 \wedge d_3) + (\alpha + 1)\, y_6 \otimes (d_9 \wedge dy_4)$
$+ \left(\alpha^2 + \alpha\right) y_6 \otimes (dx_{10} \wedge dy_5) + \left(\alpha^2 + \alpha\right) y_6 \otimes (dx_{13} \wedge dy_{11}) + (\alpha + 1)\, y_9 \otimes (dx_6 \wedge dy_4) + y_{10} \otimes (dx_3 \wedge dy_2)$
$+ y_{10} \otimes (dx_6 \wedge dy_5) + y_{13} \otimes (dx_6 \wedge dy_{11}) + y_{13} \otimes (dy_2 \wedge dy_4) + y_{14} \otimes (dx_3 \wedge dy_{11}) + y_{14} \otimes (dy_4 \wedge dy_5)$

$\deg = -4$ $\left(\alpha^3 + \alpha^2\right) h_4 \otimes (dx_5)^{\wedge 2} + \left(\alpha^3 + \alpha^2\right) x_3 \otimes (dx_2 \wedge dx_{10}) + \left(\alpha^3 + \alpha^2\right) x_3 \otimes (dx_5 \wedge dx_8)$
$+ \left(\alpha^4 + \alpha^3\right) x_3 \otimes (dx_{15} \wedge dy_9) + \left(\alpha^3 + \alpha^2\right) x_6 \otimes (dx_5 \wedge dx_{10}) + \left(\alpha^4 + \alpha^3\right) x_6 \otimes (dx_{15} \wedge dy_7)$
$+ \left(\alpha^3 + \alpha^2\right) x_7 \otimes (dx_2 \wedge dx_{12}) + \left(\alpha^3 + \alpha^2\right) x_7 \otimes (dx_5 \wedge dx_{11}) + \left(\alpha^4 + \alpha^3\right) x_7 \otimes (dx_{15} \wedge dy_6)$
$+ \left(\alpha^3 + \alpha^2\right) x_9 \otimes (dx_5 \wedge dx_{12}) + \left(\alpha^4 + \alpha^3\right) x_9 \otimes (dx_{15} \wedge dy_3) + \left(\alpha^3 + \alpha^2\right) y_2 \otimes (dx_{10} \wedge dy_3)$
$+ \left(\alpha^3 + \alpha^2\right) y_2 \otimes (dx_{12} \wedge dy_7) + \left(\alpha^2 + \alpha\right) y_5 \otimes (dx_8 \wedge dy_3) + \left(\alpha^2 + \alpha\right) y_5 \otimes (dx_{10} \wedge dy_6)$
$+ \left(\alpha^2 + \alpha\right) y_5 \otimes (dx_{11} \wedge dy_7) + \left(\alpha^2 + \alpha\right) y_5 \otimes (d(x_{12} \wedge dy_9) + \left(\alpha^2 + \alpha\right) y_8 \otimes (dx_5 \wedge dy_3)$
$+ \alpha\, y_{10} \otimes (dx_2 \wedge dy_3) + \alpha\, y_{10} \otimes (dx_5 \wedge dy_6) + \left(\alpha^2 + \alpha\right) y_{11} \otimes (dx_5 \wedge dy_7) + \alpha\, y_{12} \otimes (dx_2 \wedge dy_7)$
$+ \alpha\, y_{12} \otimes (dx_5 \wedge dy_9) + y_{15} \otimes (dy_3 \wedge dy_9) + y_{15} \otimes (dy_6 \wedge dy_7)$

$\deg = -2$ $(\alpha + 1)\, h_4 \otimes (dx_1)^{\wedge 2} + \left(\alpha^2 + \alpha\right) x_2 \otimes (d(x_{10} \wedge dy_3) + \left(\alpha^2 + \alpha\right) x_2 \otimes (dx_{12} \wedge dy_7) + (\alpha + 1)\, x_3 \otimes (dx_1 \wedge dx_6)$
$+ \left(\alpha^2 + \alpha\right) x_3 \otimes (dx_{10} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) x_3 \otimes (dx_{14} \wedge dy_{11}) + (\alpha + 1)\, x_7 \otimes (dx_1 \wedge dx_9)$
$+ \left(\alpha^2 + \alpha\right) x_7 \otimes (dx_{12} \wedge dy_2) + \left(\alpha^3 + \alpha^2\right) x_7 \otimes (d_{14} \wedge dy_8) + (\alpha + 1)\, x_8 \otimes (dx_1 \wedge dx_{10})$
$+ \left(\alpha^2 + \alpha\right) x_8 \otimes (dx_{14} \wedge dy_7) + (\alpha + 1)\, x_{11} \otimes (dx_1 \wedge dx_{12}) + \left(\alpha^2 + \alpha\right) x_{11} \otimes (dx_{14} \wedge dy_3)$
$+ (\alpha + 1)\, y_1 \otimes (dx_6 \wedge dy_3) + (\alpha + 1)\, y_1 \otimes (dx_9 \wedge dy_7) + \left(\alpha^2 + \alpha\right) y_1 \otimes (dx_{10} \wedge dy_8)$
$+ \left(\alpha^2 + \alpha\right) y_1 \otimes (dx_{12} \wedge dy_{11}) + (\alpha + 1)\, y_6 \otimes (dx_1 \wedge dy_3) + (\alpha + 1)\, y_9 \otimes (dx_1 \wedge dy_7) + y_{10} \otimes (dx_1 \wedge dy_8)$
$+ y_{10} \otimes (dy_2 \wedge dy_3) + y_{12} \otimes (dx_1 \wedge dy_{11}) + y_{12} \otimes (dy_2 \wedge dy_7) + y_{14} \otimes (dy_3 \wedge dy_{11}) + y_{14} \otimes (dy_7 \wedge dy_8)$

$\deg = 0$ $\alpha\, h_1 \otimes (d_{14} \wedge dy_{14}) + \alpha^2 h_1 \otimes (dx_{15} \wedge dy_{15}) + h_2 \otimes (dx_5 \wedge dy_5) + h_2 \otimes (dx_8 \wedge dy_8) + h_2 \otimes (dx_{10} \wedge dy_{10})$
$+ h_2 \otimes (dx_{11} \wedge dy_{11}) + h_2 \otimes (dx_{12} \wedge dy_{12}) + h_2 \otimes (dx_{13} \wedge dy_{13}) + \alpha^2 h_2 \otimes (dx_{14} \wedge dy_{14}) + h_3 \otimes (dx_1 \wedge dy_1)$
$+ \alpha\, h_3 \otimes (dx_5 \wedge dy_5) + h_3 \otimes (dx_6 \wedge dy_6) + \alpha\, h_3 \otimes (dx_8 \wedge dy_8) + h_3 \otimes (dx_9 \wedge dy_9) + \alpha\, h_3 \otimes (dx_{10} \wedge dy_{10})$
$+ \alpha\, h_3 \otimes (dx_{11} \wedge dy_{11}) + \alpha\, h_3 \otimes (dx_{12} \wedge dy_{12}) + \left(\alpha^3 + \alpha^2\right) h_3 \otimes (dx_{14} \wedge dy_{14}) + \left(\alpha^4 + \alpha^3\right) h_3 \otimes (dx_{15} \wedge dy_{15})$
$+ \alpha\, h_4 \otimes (dx_{12} \wedge dy_{12}) + \alpha\, h_4 \otimes (dx_{13} \wedge dy_{13}) + \alpha\, h_4 \otimes (dx_{14} \wedge dy_{14}) + \alpha^2 h_4 \otimes (dx_{15} \wedge dy_{15}) + x_1 \otimes (dh_1 \wedge dx_1)$
$+ \alpha\, x_1 \otimes (dx_{10} \wedge dy_8) + \alpha\, x_1 \otimes (dx_{12} \wedge dy_{11}) + \alpha\, x_1 \otimes (dx_{14} \wedge dy_{13}) + x_2 \otimes (dh_1 \wedge dx_2) + x_2 \otimes (dx_5 \wedge dy_1)$
$+ x_2 \otimes (dx_8 \wedge dy_6) + x_2 \otimes (dx_{11} \wedge dy_9) + \alpha^3 x_2 \otimes (dx_{15} \wedge dy_{14}) + \alpha\, x_3 \otimes (dx_{13} \wedge dy_{12}) + x_4 \otimes (dh_1 \wedge dx_4)$
$+ \alpha\, x_4 \otimes (dx_{12} \wedge dy_{10}) + x_5 \otimes (dx_{10} \wedge dy_6) + x_5 \otimes (dx_{12} \wedge dy_9) + \alpha\, x_5 \otimes (dx_{15} \wedge dy_{13}) + x_6 \otimes (dh_1 \wedge dx_6)$
$+ \alpha\, x_6 \otimes (dx_{10} \wedge dy_5) + \alpha\, x_6 \otimes (dx_{13} \wedge dy_{11}) + \alpha\, x_6 \otimes (dx_{14} \wedge dy_{12}) + x_7 \otimes (dh_1 \wedge dx_7) + \alpha\, x_7 \otimes (dx_{13} \wedge dy_{10})$
$+ x_8 \otimes (dx_{10} \wedge dy_1) + x_8 \otimes (dx_{13} \wedge dy_9) + \alpha\, x_8 \otimes (dx_{15} \wedge dy_{12}) + \alpha\, x_9 \otimes (dx_{12} \wedge dy_5) + \alpha\, x_9 \otimes (dx_{13} \wedge dy_8)$
$+ \alpha\, x_9 \otimes (dx_{14} \wedge dy_{10}) + x_{10} \otimes (dh_1 \wedge dx_{10}) + x_{10} \otimes (dx_{14} \wedge dy_9) + \alpha\, x_{10} \otimes (dx_{15} \wedge dy_{11}) + x_{11} \otimes (dh_1 \wedge dx_{11})$
$+ x_{11} \otimes (dx_{12} \wedge dy_1) + x_{11} \otimes (dx_{13} \wedge dy_6) + \alpha\, x_{11} \otimes (dx_{15} \wedge dy_{10}) + x_{12} \otimes (dx_{14} \wedge dy_6) + \alpha\, x_{12} \otimes (dx_{15} \wedge dy_8)$
$+ x_{13} \otimes (dx_{14} \wedge dy_1) + \alpha\, x_{13} \otimes (dx_{15} \wedge dy_5) + x_{14} \otimes (dh_1 \wedge dx_{14}) + y_1 \otimes (dh_1 \wedge dy_1) + \alpha\, y_1 \otimes (dx_8 \wedge dy_{10})$
$+ \alpha\, y_1 \otimes (dx_{11} \wedge dy_{12}) + \alpha\, y_1 \otimes (dx_{13} \wedge dy_{14}) + y_2 \otimes (dh_1 \wedge dy_2) + y_2 \otimes (dx_1 \wedge dy_5) + y_2 \otimes (dx_6 \wedge dy_8)$
$+ y_2 \otimes (dx_9 \wedge dy_{11}) + \alpha^3 y_2 \otimes (dx_{14} \wedge dy_{15}) + \alpha\, y_3 \otimes (dx_{12} \wedge dy_{13}) + y_4 \otimes (dh_1 \wedge dy_4) + \alpha\, y_4 \otimes (dx_{10} \wedge dy_{12})$
$+ y_5 \otimes (dx_6 \wedge dy_{10}) + y_5 \otimes (dx_9 \wedge dy_{12}) + \alpha\, y_5 \otimes (dx_{13} \wedge dy_{15}) + y_6 \otimes (dh_1 \wedge dy_6) + \alpha\, y_6 \otimes (dx_5 \wedge dy_{10})$
$+ \alpha\, y_6 \otimes (dx_{11} \wedge dy_{13}) + \alpha\, y_6 \otimes (dx_{12} \wedge dy_{14}) + y_7 \otimes (dh_1 \wedge dy_7) + \alpha\, y_7 \otimes (dx_{10} \wedge dy_{13}) + y_8 \otimes (dx_1 \wedge dy_{10})$
$+ y_8 \otimes (dx_9 \wedge dy_{13}) + \alpha\, y_8 \otimes (dx_{12} \wedge dy_{15}) + \alpha\, y_9 \otimes (dx_5 \wedge dy_{12}) + \alpha\, y_9 \otimes (dx_8 \wedge dy_{13}) + \alpha\, y_9 \otimes (dx_{10} \wedge dy_{14})$
$+ y_{10} \otimes (dh_1 \wedge dy_{10}) + y_{10} \otimes (dx_9 \wedge dy_{14}) + \alpha\, y_{10} \otimes (dx_{11} \wedge dy_{15}) + y_{11} \otimes (dh_1 \wedge dy_{11}) + y_{11} \otimes (dx_1 \wedge dy_{12})$
$+ y_{11} \otimes (dx_6 \wedge dy_{13}) + \alpha\, y_{11} \otimes (dx_{10} \wedge dy_{15}) + y_{12} \otimes (dx_6 \wedge dy_{14}) + \alpha\, y_{12} \otimes (dx_8 \wedge dy_{15}) + y_{13} \otimes (dx_1 \wedge dy_{14})$
$+ \alpha\, y_{13} \otimes (dx_5 \wedge dy_{15}) + y_{14} \otimes (dh_1 \wedge dy_{14})$

4.9. $\mathfrak{gl}(4)$ and its simple relative. The Lie algebra [4) $\mathfrak{g} = \mathfrak{psl}(4)$ has no Cartan matrix; its relative that has a Cartan matrix which is (45) is $\mathfrak{gl}(4)$.

Lemma. *In order to compare with Shen's "variations of $\mathfrak{g}(2)$ theme" we give, in parentheses, together with the generators of $\mathfrak{sl}(4)$, their weights in terms of the root systems of $\mathfrak{gl}(4)$ (β's) and $\mathfrak{g}(2)$ (α's):*

(45)
$$\begin{pmatrix} \bar{0} & 1 & 0 \\ 1 & \bar{0} & 1 \\ 0 & 1 & \bar{0} \end{pmatrix}$$

$x_1 = E_{12}$ $(\beta_1 = \alpha_2)$, $\quad x_2 = E_{23}$ $(\beta_2 = \alpha_1)$, $\quad x_3 = E_{34}$ $(\beta_3 = 2\alpha_1 + \alpha_2)$,
$x_4 := [x_1, x_2] = E_{13}$ $(\beta_1 + \beta_2 = \alpha_1 + \alpha_2)$, $\quad x_5 := [x_2, x_3] = E_{24}$ $(\beta_2 + \beta_3 = 3\alpha_1 + \alpha_2)$,
$x_6 := [x_3, [x_1, x_2]] = E_{14}$ $(\beta_1 + \beta_2 + \beta_3 = 3\alpha_1 + 2\alpha_2)$

Then $H^2(\mathfrak{g}; \mathfrak{g})$ is spanned by the cocycles (46) and (47). The parameters a, b in parentheses near the degree correspond to Shen's parameters of his $V_4\mathfrak{g}(2; a, b)$.

(46)

$\deg = -6 \qquad -4(\alpha_1 + \alpha_2) \qquad y_1 \otimes dx_4 \wedge dx_6 + y_4 \otimes dx_1 \wedge dx_6 + y_6 \otimes dx_1 \wedge dx_4$

$\deg = -6 \qquad -4(2\alpha_1 + \alpha_2) \qquad y_3 \otimes dx_5 \wedge dx_6 + y_5 \otimes dx_3 \wedge dx_6 + y_6 \otimes dx_3 \wedge dx_5$

$\deg = -4 \ (b) \qquad -2(2\alpha_1 + \alpha_2) \qquad h_2 \otimes dx_4 \wedge dx_5 + y_2 \otimes dh_3 \wedge dx_6 + y_2 \otimes dx_1 \wedge x_5 + y_2 \otimes dx_3 \wedge dx_4 + y_4 \otimes dh_3 \wedge dx_5 +$
$\qquad \qquad \qquad + y_5 \otimes dh_3 \wedge dx_4 + y_6 \otimes dh_3 \wedge dx_2 + y_6 \otimes dx_4 \wedge dy_1 + y_6 \otimes dx_5 \wedge dy_3$

$\deg = -4 \qquad -3(\alpha_1 + \alpha_2) \qquad h_3 \otimes dx_2 \wedge dx_6 + x_1 \otimes dx_4 \wedge dx_6 + x_3 \otimes dx_5 \wedge dx_6 + y_2 \otimes dh_2 \wedge dx_6$
$\qquad \qquad \qquad + y_2 \otimes dh_3 \wedge dx_6 + y_2 \otimes dx_1 \wedge dx_5 + y_2 \otimes dx_3 \wedge dx_4 + y_4 \otimes dh_2 \wedge dx_5 +$
$\qquad \qquad \qquad y_4 \otimes dh_3 \wedge dx_5 + y_4 \otimes dx_2 \wedge dx_3 + y_5 \otimes dh_2 \wedge dx_4 + y_5 \otimes dh_3 \wedge dx_4$
$\qquad \qquad \qquad + y_5 \otimes dx_1 \wedge dx_2 + y_6 \otimes dh_2 \wedge dx_2 + y_6 \otimes dh_3 \wedge dx_2$

(47)

$\deg = -2 \qquad 0 \qquad x_3 \otimes dx_2 \wedge dx_4 + y_2 \otimes dx_4 \wedge dy_3 + y_4 \otimes dx_2 \wedge dy_3$

$\deg = -2 \qquad -4\alpha_1 \qquad x_1 \otimes dx_2 \wedge dx_5 + y_2 \otimes dx_5 \wedge dy_1 + y_5 \otimes dx_2 \wedge dy_1$

$\deg = -2 \qquad -2(\alpha_1 + \alpha_2) \qquad h_3 \otimes dx_6 \wedge dy_2 + x_2 \otimes dh_2 \wedge dx_6 + y_1 \otimes dh_2 \wedge dx_3 + y_1 \otimes dx_5 \wedge dy_2 + y_1 \otimes dx_6 \wedge dy_4$

$\qquad \qquad \qquad + y_3 \otimes dh_2 \wedge dx_1 + y_3 \otimes dx_4 \wedge dy_2 + y_3 \otimes dx_6 \wedge dy_5 + y_6 \otimes dh_2 \wedge dy_2$

$\deg = -2 \qquad -2(\alpha_1 + \alpha_2) \qquad h_2 \otimes dx_1 \wedge dx_3 + x_2 \otimes dh_3 \wedge dx_6 + x_4 \otimes dx_1 \wedge dx_6 + x_5 \otimes dx_3 \wedge dx_6 + y_1 \otimes dh_3 \wedge dx_3$

$\qquad \qquad \qquad + y_1 \otimes dx_5 \wedge dy_2 + y_1 \otimes dx_6 \wedge dy_4 + y_3 \otimes dh_3 \wedge dx_1 + y_3 \otimes dx_4 \wedge dy_2 + y_3 \otimes dx_6 \wedge dy_5 +$
$\qquad \qquad \qquad y_4 \otimes dx_3 \wedge dy_2 + y_5 \otimes dx_1 \wedge dy_2 + y_6 \otimes dh_3 \wedge dy_2$

$\deg = 0 \qquad 2\alpha_1 \qquad h_3 \otimes dx_4 \wedge dy_5 + x_3 \otimes dh_2 \wedge dx_1 + x_3 \otimes dx_4 \wedge dy_2 + x_3 \otimes dx_6 \wedge dy_5 + x_5 \otimes dh_2 \wedge dx_4$

$\qquad \qquad \qquad + y_1 \otimes dh_2 \wedge dy_3 + y_1 \otimes dx_2 \wedge dy_5 + y_1 \otimes dx_4 \wedge dy_6 + y_4 \otimes dh_2 \wedge dy_5$

$\deg = 0 \qquad -2\alpha_1 \qquad h_3 \otimes dx_5 \wedge dy_4 + x_1 \otimes dh_2 \wedge dx_3 + x_1 \otimes dx_5 \wedge dy_2 + x_1 \otimes dx_6 \wedge dy_4 + x_4 \otimes dh_2 \wedge dx_5$

$\qquad \qquad \qquad + y_3 \otimes dh_2 \wedge dy_1 + y_3 \otimes dx_2 \wedge dy_4 + y_3 \otimes dx_5 \wedge dy_6 + y_5 \otimes dh_2 \wedge dy_4$

$\deg = 0 \ (a) \qquad 2\alpha_1 \qquad h_2 \otimes dx_1 \wedge dy_3 + x_2 \otimes dx_4 \wedge dy_3 + x_3 \otimes dh_2 \wedge dx_1 + x_3 \otimes dh_3 \wedge dx_1 + x_5 \otimes dh_2 \wedge dx_4$

$\qquad \qquad \qquad + x_5 \otimes dh_3 \wedge dx_4 + x_5 \otimes dx_1 \wedge dx_2 + y_1 \otimes dh_2 \wedge dy_3 + y_1 \otimes dh_3 \wedge dy_3 + y_1 \otimes dx_2 \wedge dy_5 +$
$\qquad \qquad \qquad y_1 \otimes dx_4 \wedge dy_6 + y_4 \otimes dh_2 \wedge dy_5 + y_4 \otimes dh_3 \wedge dy_5 + y_4 \otimes dx_1 \wedge dy_6 + y_6 \otimes dy_3 \wedge dy_5$

$\deg = 0 \qquad -2\alpha_1 \qquad h_2 \otimes dx_3 \wedge dy_1 + x_1 \otimes dh_2 \wedge dx_3 + x_1 \otimes dh_3 \wedge dx_3 + x_2 \otimes dx_5 \wedge dy_1 + x_4 \otimes dh_2 \wedge dx_5$

$\qquad \qquad \qquad + x_4 \otimes dh_3 \wedge dx_5 + x_4 \otimes dx_2 \wedge dx_3 + y_3 \otimes dh_2 \wedge dy_1 + y_3 \otimes dh_3 \wedge dy_1 + y_3 \otimes dx_2 \wedge dy_4 +$
$\qquad \qquad \qquad y_3 \otimes dx_5 \wedge dy_6 + y_5 \otimes dh_2 \wedge dy_4 + y_5 \otimes dh_3 \wedge dy_4 + y_5 \otimes dx_3 \wedge dy_6 + y_6 \otimes dy_1 \wedge dy_4$

[4)]Observe that $\mathfrak{psl}(4) := \mathfrak{g}(2)$ in the basis given in [FH], p. 346. Shen himself observed this, and therefore the property of the root systems to have all roots of "equal length" (whatever that might mean if $p = 2$) is not an invariant if $p = 2$.

5. On integrability

Consider the bracket

$$[\cdot,\cdot]_c = [\cdot,\cdot] + tc. \tag{48}$$

5.1. Lemma. *For $p = 3$, the bracket (48) satisfies the Jacobi identity in the following cases:*
  1) For $\mathfrak{br}(2)$, $\mathfrak{brj}(2,3)$, $\mathfrak{br}(2,5)$ and $\mathfrak{br}(3)$, for every of the above homogeneous cocycles c;
  2) For $\mathfrak{g}(2,3)$, never;
  3) For $\mathfrak{g}(1,6)$, only for the cocycle c of degree -12.

5.1.1. Remark. Although JI is not satisfied for the bracket (48) for some cocycles c, it does not mean that these cocycles are not integrable: The Jacobi identity might be satisfied by a bracket of the form $[\cdot,\cdot]_c = [\cdot,\cdot] + tc_1 + t^2 c_2 + \dots$. We have started to investigate this option elsewhere, together with integrability study of multiparameter deformations, except for the simplest cases where the global deform is linear in all parameters simultaneously.

5.2. Lemma. *For $p = 2$, the bracket (48) satisfies the Jacobi identity in the following cases:*
  1) For $\mathfrak{o}^{(1)}(3)$, $\mathfrak{o}^{(1)}(5)$, $\mathfrak{psl}(4)$, and $\mathfrak{psl}(6)$, for every of the above homogeneous cocycles c.
  2) For $\mathfrak{wk}(3;\alpha)$ and $\mathfrak{wk}(4;\alpha)$, and also for $\mathfrak{bgl}(3;\alpha)$ and $\mathfrak{bgl}(4;\alpha)$, for every of the above homogeneous cocycles c, except the ones of degree 0.
  5) For $\mathfrak{oo}_{I\Pi}^{(1)}(1|2)$, for every of the above homogeneous cocycles c.

5.3. Lemma. *For $p = 2$, the Lie algebra $\mathfrak{o}_I^{(1)}(3)$ admits the global deformation given by*

$$[\cdot,\cdot]_{\alpha,\beta} = [\cdot,\cdot] + \alpha c_1 + \beta c_2.$$

Denote the deform by $\mathfrak{o}_I^{(1)}(3,\alpha,\beta)$. This Lie algebra is simple if and only if $\alpha\beta \neq 1$.
  If $\alpha\beta = 1$, then $\mathfrak{o}_I^{(1)}(3,\alpha,\beta)$ has an ideal $I = \operatorname{Span}\{h, x + \alpha y\}$.
  For $\alpha\beta \neq 1$, the Lie algebra $\mathfrak{o}_I^{(1)}(3,\alpha,\beta)$ is simple with two outer derivations given by

$$\alpha h \otimes dh + \alpha x \otimes dx + x \otimes dy, \quad \beta h \otimes dh + \beta x \otimes dx + y \otimes dx.$$

Proof of the statements of this Lemma are straightforward.

Reducing the operator ad_h to the Jordan normal form, we immediately see (in the eigenbasis) that the deform depends, actually, on one parameter, not two; and all of them are non-isomorphic.

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